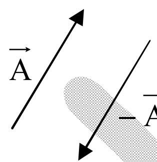
1. Vector Analysis

1.1 Vector Algebra

Vector quantities have both *direction* as well as *magnitude* such as velocity, acceleration, force and momentum etc. We will use $\vec{A}$ for any general vector and its magnitude by $|\vec{A}|$. In diagrams vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. Minus $\vec{A}\,(-\vec{A})$ is a vector with the same magnitude as $\vec{A}$ but of opposite direction.



1.1.1 Vector Operations

We define four vector operations: addition and three kinds of multiplication.

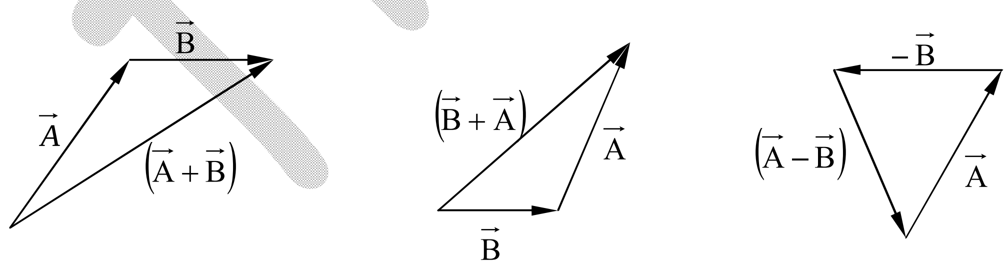
(i) Addition of two vectors

Place the tail of $\vec{B}$ at the head of $\vec{A}$; the sum, $\vec{A}+\vec{B}$, is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$.

Addition is *commutative*: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

Addition is associative: $\left(\vec{A}+\vec{B}\right)+\vec{C}=\vec{A}+\left(\vec{B}+\vec{C}\right)$

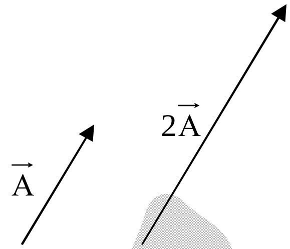
To subtract a vector, add its opposite: $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$



(ii) Multiplication by scalar

Multiplication of a vector by a positive scalar a, multiplies the *magnitude* but leaves the direction unchanged. (If a is negative, the direction is reversed.) Scalar multiplication is distributive:

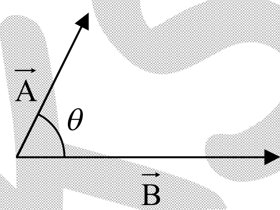
$$a\left(\vec{A}+\vec{B}\right)=a\vec{A}+a\vec{B}$$



(iii) Dot product of two vectors

The dot product of two vectors is define by

$$\vec{A}.\vec{B}=AB\cos\theta$$



where θ is the angle they form when placed tail to tail. Note that $\vec{A}.\vec{B}$ is itself a scalar. The dot product is commutative,

$$\vec{A}.\vec{B}=\vec{B}.\vec{A}$$

and distributive, $\vec{A}.\left(\vec{B}+\vec{C}\right)=\vec{A}.\vec{B}+\vec{A}.\vec{C}$

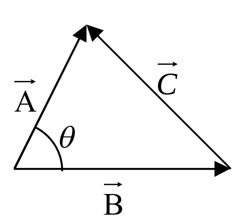
Geometrically $\vec{A}.\vec{B}$ is the product of A times the projection of $\vec{B}$ along $\vec{A}$ (or the product of B times the projection of $\vec{A}$ along $\vec{B}$).

If the two vectors are parallel, $\vec{A}.\vec{B}=AB$

If two vectors are perpendicular, then $\vec{A}.\vec{B}=0$

Law of cosines

Let $\vec{C}=\vec{A}-\vec{B}$ and then calculate dot product of $\vec{C}$ with itself.



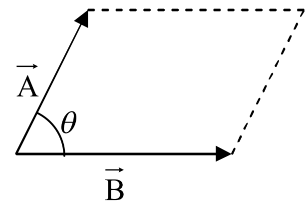
$$\vec{C}.\vec{C}=\left(\vec{A}-\vec{B}\right).\left(\vec{A}-\vec{B}\right)=\vec{A}.\vec{A}-\vec{A}.\vec{B}-\vec{B}.\vec{A}+\vec{B}.\vec{B}$$

$$C^2=A^2+B^2-2AB\cos\theta$$

(iv) Cross product of two vectors

The cross product of two vectors is define by

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$



where $\hat{n}$ is a unit vector(vector of length 1) pointing perpendicular to the plane of $\vec{A}$ and $\vec{B}$. Of course there are two directions perpendicular to any plane "in" and "out."

The ambiguity is resolved by the **right-hand rule:**

let your fingers point in the direction of first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{n}$. (In figure $\vec{A}\times\vec{B}$ points into the page; $\vec{B}\times\vec{A}$ points out of the page)

The cross product is distributive, $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\times\vec{B}\right) + \left(\vec{A}\times\vec{C}\right)$

but not commutative. In fact, $\left(\vec{B}\times\vec{A}\right) = -\left(\vec{A}\times\vec{B}\right)$.

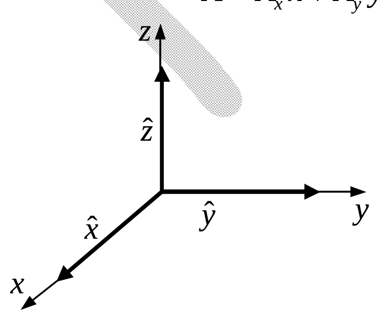
Geometrically, $\left|\vec{A}\times\vec{B}\right|$ is the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$. If two vectors are parallel, their cross product is zero.

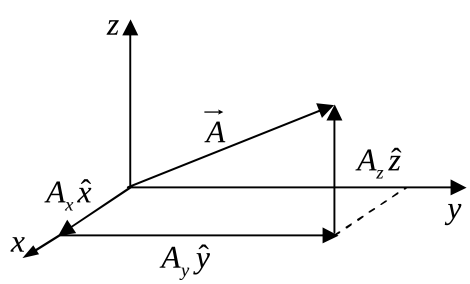
In particular $\vec{A}\times\vec{A} = 0$ for any vector $\vec{A}$

1.1.2 Vector Algebra: Component Form

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the x, y and z axis, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$





The numbers A_x, A_y, and A_z are called component of $\vec{A}$; geometrically, they are the projections of $\vec{A}$ along the three coordinate axes.

(i) Rule: To add vectors, add like components.

$$\vec{A}+\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)+\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)=\left(A_x+B_x\right)\hat{x}+\left(A_y+B_y\right)\hat{y}+\left(A_z+B_z\right)\hat{z}$$

(ii) Rule: To multiply by a scalar, multiply each component.

$$\vec{A}=\left(aA_x\right)\hat{x}+\left(aA_y\right)\hat{y}+\left(aA_z\right)\hat{z}$$

Because $\hat{x}$, $\hat{y}$ and $\hat{z}$ are mutually perpendicular unit vectors

$$\hat{x}.\hat{x}=\hat{y}.\hat{y}=\hat{z}.\hat{z}=1;\;\; \hat{x}.\hat{y}=\hat{x}.\hat{z}=\hat{y}.\hat{z}=0$$

Accordingly, $\vec{A}.\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right).\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)=A_xB_x+A_yB_y+A_zB_z$

(iii) Rule: To calculate the dot product, multiply like components, and add.

In particular, $\vec{A}.\vec{A}=A_x^2+A_y^2+A_z^2 \Rightarrow A=\sqrt{A_x^2+A_y^2+A_z^2}$

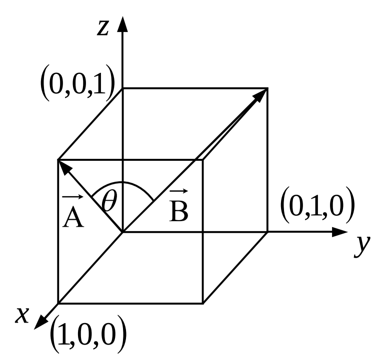
Similarly, $\hat{x}\times\hat{x}=\hat{y}\times\hat{y}=\hat{z}\times\hat{z}=0,$

$$\hat{x}\times\hat{y}=-\hat{y}\times\hat{x}=\hat{z}$$
$$\hat{y}\times\hat{z}=-\hat{z}\times\hat{y}=\hat{x}$$
$$\hat{z}\times\hat{x}=-\hat{x}\times\hat{z}=\hat{y}$$

(iv) Rule: To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\vec{A}$ (in component form), and whose third row is $\vec{B}$.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}$$

Example: Find the angle between the face diagonals of a cube.



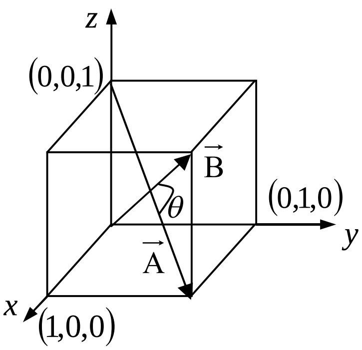
Solution: The face diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A}=1\hat{x}+0\hat{y}+1\hat{z};\quad \vec{B}=0\hat{x}+1\hat{y}+1\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B}=1$

Also, $\Rightarrow \vec{A}.\vec{B}=AB\cos\theta=\sqrt{2}\sqrt{2}\cos\theta \Rightarrow \cos\theta=\frac{1}{2}\Rightarrow \theta=60^0$

Example: Find the angle between the body diagonals of a cube.



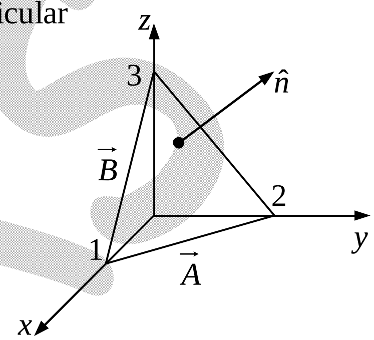
Solution: The body diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A} = \hat{x} + \hat{y} - \hat{z}; \quad \vec{B} = \hat{x} + \hat{y} + \hat{z}$

So, $\Rightarrow \vec{A}.\vec{B} = 1 + 1 - 1 = 1$

Also, $\Rightarrow \vec{A}.\vec{B} = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta \Rightarrow \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$

Example: Find the components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure.



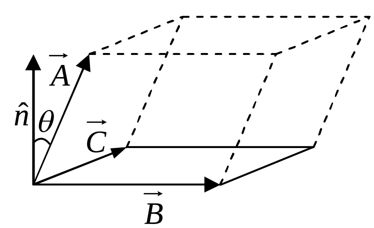
Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$\vec{A} = -\hat{x} + 2\hat{y}; \quad \vec{B} = -\hat{x} + 3\hat{z} \Rightarrow \hat{n} = \frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{7}$

1.1.3 Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product.

(i) Scalar triple product: $\vec{A}.\left(\vec{B}\times\vec{C}\right)$



Geometrically $\left|\vec{A}.\left(\vec{B}\times\vec{C}\right)\right|$ is the volume of the parallelepiped generated by $\vec{A}$, $\vec{B}$ and $\vec{C}$, since $\left|\vec{B}\times\vec{C}\right|$ is the area of the base, and $\left|\vec{A}\cos\theta\right|$ is the altitude. Evidently,

$$\vec{A}.\left(\vec{B}\times\vec{C}\right) = \vec{B}.\left(\vec{C}\times\vec{A}\right) = \vec{C}.\left(\vec{A}\times\vec{B}\right)$$

In component form $\vec{A}.\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

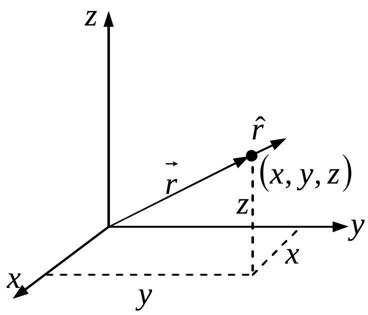
Note that the dot and cross can be interchanged: $\vec{A}.\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\times\vec{B}\right).\vec{C}$

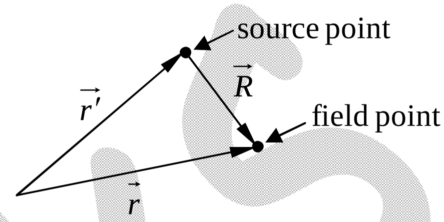
(ii) Vector triple product: $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

1.1.4 Position, Separation and Displacement Vectors





The location of a point in three dimensions can be described by listing its Cartesian coordinates(x, y, z). The vector to that point from the origin is called the ***position vector***:

$$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}.$$

Its magnitude, $r=\sqrt{x^2+y^2+z^2}$ is the distance from the origin,

and $\hat{r}=\dfrac{\vec{r}}{r}=\dfrac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$ is a unit vector pointing radially outward.

Note: In electrodynamics one frequently encounters problems involving two points-typically, a **source point ,** $\vec{r}'$, where an electric charge is located, and a **field point,** $\vec{r}$ **,** at which we are calculating the electric or magnetic field. We can define **separation vector** from the source point to the field point by $\vec{R}$;

$$\vec{R}=\vec{r}-\vec{r}'.$$

Its magnitude is $$R=\left|\vec{r}-\vec{r}'\right|,$$

and a unit vector in the direction from $\vec{r}'$ to $\vec{r}$ is $\hat{R}=\dfrac{\vec{R}}{R}=\dfrac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|}$.

In Cartesian coordinates, $\vec{R}=\left(x-x'\right)\hat{x}+\left(y-y'\right)\hat{y}+\left(z-z'\right)\hat{z}$

$$\left|\vec{R}\right| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\hat{R} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

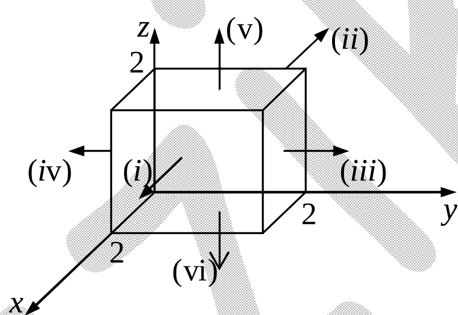
Infinitesimal Displacement Vector $\left(d\vec{l}\right)$

The infinitesimal displacement vector, from (x, y, z) to $(x+dx, y+dy, z+dz)$, is

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

Area Element $\left(d\vec{a}\right)$

For closed surface area element is perpendicular to the surface pointing outwards as shown in figure below.

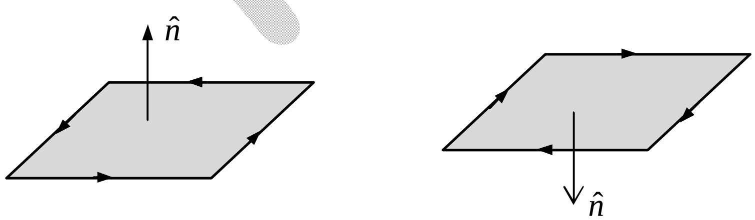


(i) For $x = 2$ plane, $d\vec{a} = dydz\hat{x}$ (ii) For $x = 0$ plane, $d\vec{a} = -dydz\hat{x}$

(iii) For $y = 2$ plane, $d\vec{a} = dxdz\hat{y}$ (iv) For $y = 0$ plane, $d\vec{a} = -dxdz\hat{y}$

(v) For $z = 2$ plane, $d\vec{a} = dxdy\hat{z}$ (vi) For $z = 0$ plane, $d\vec{a} = -dxdy\hat{z}$

For open surface area element is shown in figure below (use right hand rule)



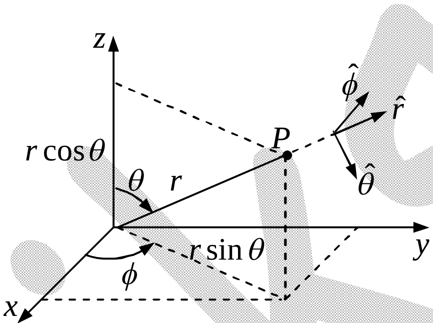
Volume Element $(d\tau)$

Volume element $d\tau = dxdydz$

1.2 Curvilinear Coordinates

1.2.1 Spherical Polar Coordinates

In spherical polar coordinate any general point P lies on the surface of a sphere. The spherical polar coordinates (r,θ,ϕ) of a point P are defined in figure shown below; r is the distance from the origin (the magnitude of the position vector), θ (the angle drawn from the z axis) is called the polar angle, and ϕ (the angle around from the x axis) is the azimuthal angle.



Their relation to Cartesian coordinates (x, y, z) can be read from the figure:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and $r = \sqrt{x^2+y^2+z^2}$, $\theta = \cos^{-1}\left(\frac{z}{r}\right)$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

The range of r is $0 \to \infty$, θ goes from $0 \to \pi$, and ϕ goes from $0 \to 2\pi$.

Figure shows three unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{x}, \hat{y}, \hat{z}$), and any vector $\vec{A}$ can be expressed in terms of them in the usual way:

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

A_r, A_θ, and A_ϕ are the radial, polar and azimuthal components of $\vec{A}$.

Infinitesimal Displacement Vector $\left(d\vec{l}\right)$

An infinitesimal displacement in the $\hat{r}$ direction is simply dr (figure a), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (figure b) is $r\,d\theta$

$$dl_\theta = r d\theta$$

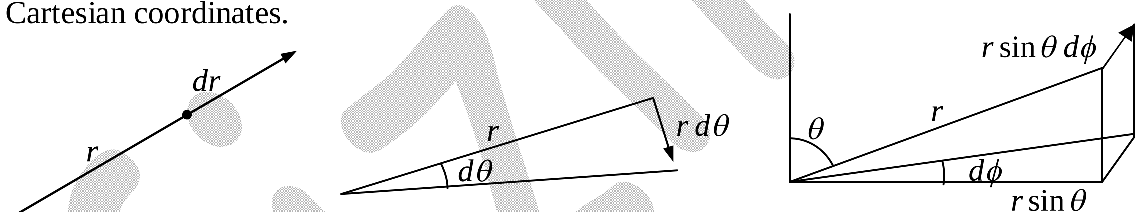
Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (figure c) is $r\sin\theta\,d\phi$

$$dl_\phi = r\sin\theta\,d\phi$$

Thus, the general infinitesimal displacement $d\vec{l}$ is

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi}$$

This plays the role (in line integrals, for example) that $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ played in Cartesian coordinates.



(a) (b) (c)

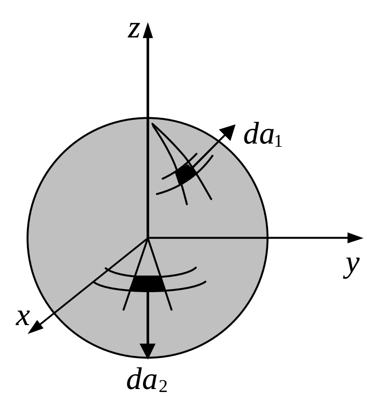
Area Element $\left(d\vec{a}\right)$

If we are integrating over the surface of a sphere, for instance, then r is constant, whereas θ and ϕ change, so

$$d\vec{a}_1 = dl_\theta dl_\phi \hat{r} = r^2 \sin\theta\,d\theta\,d\phi\,\hat{r}$$

on the other hand, if the surface lies in the xy plane, then θ is constant ($\theta = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{\theta} = r^2\,dr\,d\phi\hat{\theta}$$



Volume Element $(d\tau)$

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Transformation of Vector to Spherical Polar

We can transform any vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$ in Cartesian coordinates to Spherical polar coordinate as $\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$.

Thus

$$A_r = \vec{A}.\hat{r} = A_x(\hat{x}.\hat{r}) + A_y(\hat{y}.\hat{r}) + A_z(\hat{z}.\hat{r})$$

$$A_\theta = \vec{A}.\hat{\theta} = A_x(\hat{x}.\hat{\theta}) + A_y(\hat{y}.\hat{\theta}) + A_z(\hat{z}.\hat{\theta})$$

$$A_\phi = \vec{A}.\hat{\phi} = A_x(\hat{x}.\hat{\phi}) + A_y(\hat{y}.\hat{\phi}) + A_z(\hat{z}.\hat{\phi})$$

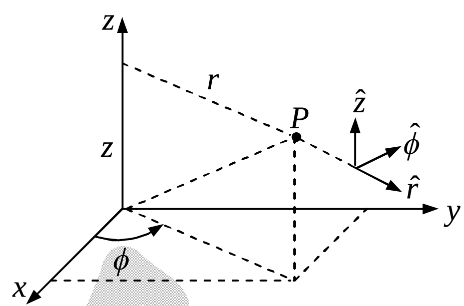
where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and use table given below:

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}.$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}.$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}.$	$\cos\theta$	$-\sin\theta$	0

1.2.2 Cylindrical Polar Coordinates

The cylindrical coordinates r, ϕ, z of a point P are defined in figure. Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; r is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is



$$x = r\cos\phi,\quad y = r\sin\phi\ ,\quad z = z$$

The range of r is $0 \to \infty$, ϕ goes from $0 \to 2\pi$, and z from $-\infty$ to ∞

The infinitesimal displacements are

$$dl_r = dr,\ \ dl_\phi = rd\phi,\ \ dl_z = dz,$$

so $$d\vec{l} = dr\,\hat{r} + r\,d\phi\hat{\phi} + dz\,\hat{z}$$

and volume element is $d\tau = r\,dr\,d\phi\,dz$.

We can transform any vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ in Cartesian coordinates to cylindrical coordinates as $\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$

Thus,

$$A_r = \vec{A}.\hat{r} = A_x(\hat{x}.\hat{r}) + A_y(\hat{y}.\hat{r}) + A_z(\hat{z}.\hat{r})$$

$$A_\phi = \vec{A}.\hat{\phi} = A_x(\hat{x}.\hat{\phi}) + A_y(\hat{y}.\hat{\phi}) + A_z(\hat{z}.\hat{\phi})$$

$$A_z = \vec{A}.\hat{z} = A_x(\hat{x}.\hat{z}) + A_y(\hat{y}.\hat{z}) + A_z(\hat{z}.\hat{z})$$

use table given below:

	$\hat{r}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}.$	$\cos\phi$	$-\sin\phi$	0
$\hat{y}.$	$\sin\phi$	$\cos\phi$	0
$\hat{z}.$	0	0	1

1.3 Differential Calculus

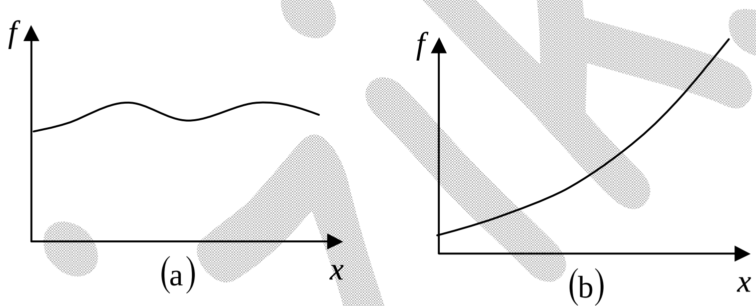
1.3.1 "Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$ then the derivative, df/dx tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right)dx$$

In words: If we change x by an amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example in figure (a), the function varies slowly with x, and the derivative is correspondingly small. In figure (b), f increases rapidly with x, and the derivative is large, as we move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the slope of the graph of f versus x.



1.3.2 Gradient

Suppose that we have a function of three variables-say, $V(x, y, z)$ in a

$$dV = \left(\frac{\partial V}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z}\right)dz.$$

This tells us how V changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do not require an infinite number of derivatives-three will suffice: the partial derivatives along each of the three coordinate directions.

Thus $dV = \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \left(\vec{\nabla}V\right)\cdot\left(d\vec{l}\right)$,

where $\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$ is the gradient of V.

$\vec{\nabla}V$ is a vector quantity, with three components.

Geometrical Interpretation of the Gradient

Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning, let's rewrite

$$dV = \vec{\nabla} V \cdot d\vec{l} = |\vec{\nabla} V||d\vec{l}|\cos\theta$$

where θ is the angle between $\vec{\nabla} V$ and $d\vec{l}$. Now, if we fix the magnitude $|d\vec{l}|$ and search around in various directions (that is, vary θ), the maximum change in V evidently occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $|d\vec{l}|$, dT is greatest when one move in the same direction as $\vec{\nabla} V$. Thus:

The gradient $\vec{\nabla} V$ points in the direction of maximum increase of the function V.

Moreover:

The magnitude $|\vec{\nabla} V|$ gives the slope (rate of increase) along this maximal direction.

Gradient in Spherical polar coordinates $V(r,\theta,\phi)$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Gradient in cylindrical coordinates $V(r,\phi,z)$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

Example: Find the gradient of a scalar function of position V where $V(x,y,z) = x^2 y + e^z$. Calculate the magnitude of gradient at point $P(1, 5, -2)$.

Solution: $V(x,y,z) = x^2 y + e^z$

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} = 2xy\hat{x} + x^2\hat{y} + e^z\hat{z}$$

At $P(1, 5, -2) \Rightarrow \vec{\nabla} V = 10\hat{x} + \hat{y} + 0.1353\hat{z} \Rightarrow |\vec{\nabla} V| = \sqrt{10^2 + 1^2 + 0.1353^2} = 10.056$

Example: Find the unit vector normal to the curve $y = x^2$ at the point (2, 4, 1).

Solution: The equation of curve in the form of surface is given by

$$x^2 - y = 0$$

A constant scalar function V on the surface is given by $V(x, y, z) = x^2 - y$

Taking the gradient

$$\vec{\nabla}V = \vec{\nabla}\left(x^2 - y\right) = \frac{\partial}{\partial x}\left(x^2 - y\right)\hat{x} + \frac{\partial}{\partial y}\left(x^2 - y\right)\hat{y} + \frac{\partial}{\partial z}\left(x^2 - y\right)\hat{z} = 2x\hat{x} - \hat{y}$$

The value of the gradient at point (2, 4, 1), $\vec{\nabla}V = 4\hat{x} - \hat{y}$

The unit vector, as required

$$\hat{n} = \pm\frac{4\hat{x} - \hat{y}}{\left|4\hat{x} - \hat{y}\right|} = \pm\frac{1}{\sqrt{17}}\left(4\hat{x} - \hat{y}\right)$$

Example: Find the unit vector normal to the surface $xy^3z^2 = 4$ at a point (-1, -1, 2).

Solution:

$$\vec{\nabla}\left(xy^3z^2\right) = \frac{\partial}{\partial x}\left(xy^3z^2\right)\hat{x} + \frac{\partial}{\partial y}\left(xy^3z^2\right)\hat{y} + \frac{\partial}{\partial z}\left(xy^3z^2\right)\hat{z} = \left(y^3z^2\right)\hat{x} + \left(3xy^2z^2\right)\hat{y} + \left(2xy^3z\right)\hat{z}$$

At a point (-1, -1, 2), $\vec{\nabla}\left(xy^3z^2\right) = -4\hat{x} - 12\hat{y} + 4\hat{z}$

Unit vector normal to the surface

$$\hat{n} = \frac{-4\hat{x} - 12\hat{y} + 4\hat{z}}{\sqrt{(-4)^2 + \left(-12^2\right) + 4^2}} = -\frac{1}{\sqrt{11}}\left(\hat{x} + 3\hat{y} - \hat{z}\right)$$

Example: In electrostatic field problems, the electric field is given by $\vec{E} = -\vec{\nabla}V$, where V is the scalar field potential. If $V = r^2\phi - 2\theta$ in spherical coordinates, then find $\vec{E}$.

Solution: $V = r^2\phi - 2\theta$

In spherical coordinate, $\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$

Substituting the suitable values, $\vec{\nabla}V = 2r\phi\hat{r} - \frac{2}{r}\hat{\theta} + \frac{r^2}{r\sin\theta}\hat{\phi}$

$$\Rightarrow \vec{E} = -\vec{\nabla}V = -2r\phi\hat{r} + \frac{2}{r}\hat{\theta} - \frac{r}{\sin\theta}\hat{\phi}$$

1.3.3 The Operator $\vec{\nabla}$

The gradient has the formal appearance of a vector, $\vec{\nabla}$, "multiplying" a scalar *V*:

$$\vec{\nabla}V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)V$$

The term in parentheses is called "del":

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

We should say that $\vec{\nabla}$ is a vector operator that acts upon *V*, not a vector that multiplies *V*.

There are three ways the operator $\vec{\nabla}$ can act:

1. on a scalar function *V*: $\vec{\nabla}V$ (the **gradient**);
2. on a vector function $\vec{A}$, via the dot product: $\vec{\nabla}\cdot\vec{A}$ (the **divergence**);
3. on a vector function $\vec{A}$, via the cross product: $\vec{\nabla}\times\vec{A}$ (the **curl**).

1.3.4 The Divergence

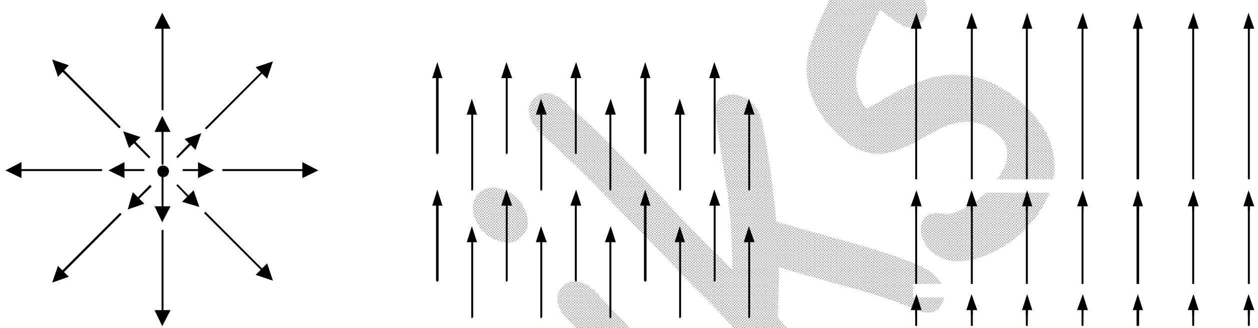
From the definition of $\vec{\nabla}$ we construct the divergence:

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Observe that the divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla}\cdot\vec{A}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation

$\vec{\nabla}\cdot\vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence. (Please understand that $\vec{A}$ here is a function-there's a different vector associated with every point in space.)



(a) (b) (c)

Divergence in Spherical polar coordinates

$$\vec{\nabla}.\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Divergence in cylindrical coordinates

$$\vec{\nabla}.\vec{A}=\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$$

Example: Suppose the function sketched in above figure are $\vec{A}=x\hat{x}+y\hat{y}+z\hat{z}$, $\vec{B}=\hat{z}$ and $\vec{C}=z\hat{z}$. Calculate their divergences.

Solution:
$$\vec{\nabla}\cdot\vec{A}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

$$\vec{\nabla}\cdot\vec{B}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(1)=0+0+0=0$$

$$\vec{\nabla}\cdot\vec{C}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(z)=0+0+1=1$$

Example: Given

(i) $\vec{A} = 2xy\hat{x} + z\hat{y} + yz^2\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$ at $(2,-1,3)$

(ii) $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$

(iii) $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$, Find $\vec{\nabla}\cdot\vec{A}$

Solution: (i) In Cartesian coordinates $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$A_x = 2xy, A_y = z, A_z = yz^2 \Rightarrow \vec{\nabla}\cdot\vec{A} = 2y + 0 + 2yz$, At $(2,-1,3)$, $\vec{\nabla}\cdot\vec{A} = -2-6 = -8$

(ii) In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$$A_r = 2r\cos^2\phi, A_\phi = 3r^2\sin z, A_z = 4z\sin^2\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4\left(\cos^2\phi + \sin^2\phi\right) = 4$$

(iii) In spherical coordinates, $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = 10, A_\theta = 5\sin\theta, A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = (2+\cos\theta)(10/r)$$

1.3.5 The Curl

From the definition of $\vec{\nabla}$ we construct the curl

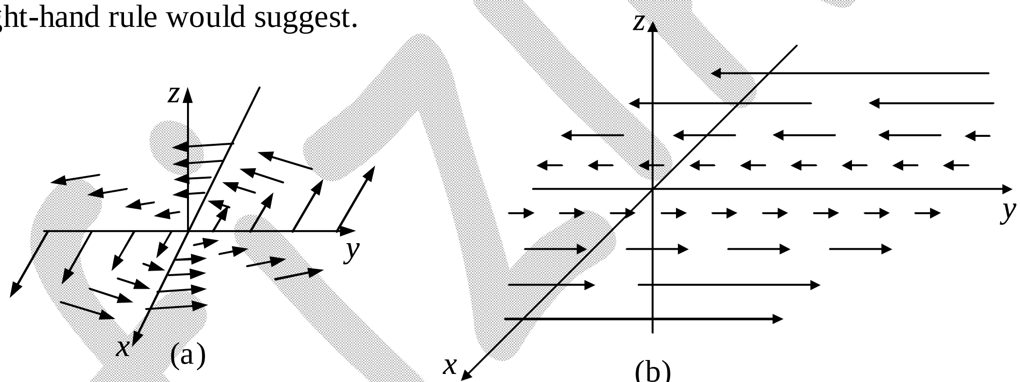
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{A}$ is, like any cross product, a vector. (You cannot have the curl of a scalar; that's meaningless.)

Geometrical Interpretation

$\vec{\nabla}\times\vec{A}$ is a measure of how much the vector $\vec{A}$ "curls around" the point in question. Figure shown below have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.



Curl in Spherical polar coordinates $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

Curl in cylindrical coordinates $\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$

Example: Suppose the function sketched in above figure are $\vec{A} = -y\hat{x} + x\hat{y}$ and $\vec{B} = x\hat{y}$. Calculate their curls.

Solution: $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{z}$ and $\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{z}$

As expected, these curls point in the +z direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"…. it just "curls around.")

Example: Given a vector function $\vec{A} = (x + c_1 z)\hat{x} + (c_2 x - 3z)\hat{y} + (x + c_3 y + c_4 z)\hat{z}$.

(a) Calculate the value of constants c_1, c_2, c_3 if $\vec{A}$ is irrotational.

(b) Determine the constant c_4 if $\vec{A}$ is also solenoidal.

(c) Determine the scalar potential function *V*, whose negative gradient equals $\vec{A}$.

Solution: If $\vec{A}$ is irrotational then, $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + c_1 z) & (c_2 x - 3z) & (x + c_3 y + c_4 z) \end{vmatrix} = 0$

$$\Rightarrow \vec{\nabla} \times \vec{A} = (c_3 + 3)\hat{x} - (1 - c_1)\hat{y} + (c_2 - 0)\hat{z} = 0 \Rightarrow c_1 = 1, c_2 = 0, c_3 = -3$$

(b) If $\vec{A}$ is solenoidal, $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1 + 0 + c_4 = 0 \Rightarrow c_4 = -1$

(c) $\vec{A} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$

$$\vec{A} = (x + z)\hat{x} + (-3z)\hat{y} + (x - 3y - z)\hat{z} \Rightarrow \frac{\partial V}{\partial x} = -x - z \Rightarrow \mathrm{V} = -\frac{x^2}{2} - xz + f_1(y, z),$$

$$\frac{\partial V}{\partial y} = 3z \Rightarrow \mathrm{V} = 3\mathrm{yz} + f_2(x, z),\ \frac{\partial V}{\partial z} = -x + 3y + z \Rightarrow V = -xz + 3yz + \frac{z^2}{2} + f_3(x, y)$$

Examination of above expressions of *V* gives a general value of

$$V = -\frac{x^2}{2} - xz + 3yz + \frac{z^2}{2}$$

Example: Find the curl of the vector $\vec{A} = \left(e^{-r}/r\right)\hat{\theta}$

Solution: $\vec{A} = \left(e^{-r}/r\right)\hat{\theta} \Rightarrow A_r = 0,\ A_\theta = \left(e^{-r}/r\right),\ A_\phi = 0$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = -\frac{e^{-r}}{r}\hat{\phi}$$

Example: Find the nature of the following fields by determining divergence and curl.

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z}$

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ (Cylindrical coordinates)

Solution:

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z} \Rightarrow \vec{\nabla}\cdot\vec{F}_1 = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 2x\left(1+5z\right)$

Divergence exists, so the field is non-solenoidal.

$$\vec{\nabla}\times\vec{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix} = -5z^2\hat{y} + 2y\hat{z}$$ .The field has a curl so it is rotational.

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ in cylindrical coordinates.

In cylindrical coordinates, Divergence $\vec{\nabla}\cdot\vec{F}_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_{2r}\right) + \frac{1}{r}\frac{\partial F_{2\phi}}{\partial \phi} + \frac{\partial F_{2z}}{\partial z} = \frac{-150}{r^3}$

The field is non-solenoid.

$$\vec{\nabla}\times\vec{F}_2 = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \left(\frac{150}{r^2}\right) & 10r & 0 \end{vmatrix} = \frac{10}{r}\hat{z}$$. $\vec{F}_2$ has non-zero curl so it is rotational.

1.3.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as

the sum rule: $\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$,

the rule for multiplying by a constant: $\frac{d}{dx}(kf)=k\frac{df}{dx}$,

the product rule: $\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$,

and the quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$.

Similar relations hold for the vector derivatives. Thus,

$$\vec{\nabla}(f+g)=\vec{\nabla}f+\vec{\nabla}g,\quad \vec{\nabla}\cdot\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{\nabla}\cdot\vec{B}\right),$$

$$\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{\nabla}\times\vec{B}\right),$$

and

$$\vec{\nabla}(kf)=k\vec{\nabla}f,\quad \vec{\nabla}\cdot\left(k\vec{A}\right)=k\left(\vec{\nabla}\cdot\vec{A}\right),\qquad \vec{\nabla}\times\left(k\vec{A}\right)=k\left(\vec{\nabla}\times\vec{A}\right),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

fg (product of two scalar functions),

$\vec{A}\cdot\vec{B}$ (Dot product of two vectors),

and two ways to make a vector:

$f\vec{A}$ (Scalar time's vector),

$\vec{A}\times\vec{B}$ (Cross product of two vectors),

Accordingly, there are six product rules,

Two for gradients

(i) $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f$,

(ii) $\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}$,

Two for divergences

(iii) $\vec{\nabla}\cdot\left(f\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right),$

(iv) $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right),$

And two for curls

(v) $\vec{\nabla}\times\left(f\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right),$

(vi) $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right),$

It is also possible to formulate three quotient rules:

$$\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2},\quad \vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}g\right)}{g^2},\quad \vec{\nabla}\times\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}g\right)}{g^2}.$$

1.3.7 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\vec{\nabla}$; by applying $\vec{\nabla}$ twice we can construct five species of second derivatives. The gradient $\vec{\nabla}V$ is a vector, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\vec{\nabla}\cdot\left(\vec{\nabla}V\right)$

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}\right)=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}.$$

This object, which we write $\nabla^2 V$ for short, is called the **Laplacian** of V. Notice that the Laplacian of a *scalar* V is a *scalar*.

Laplacian in Spherical polar coordinates

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)$$

Laplacian in cylindrical coordinates

$$\nabla^2 V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}$$

Occasionally, we shall speak of the Laplacian of a *vector*, $\nabla^2\vec{A}$. By this we mean a *vector* quantity whose x-component is the Laplacian of A_x, and so on:

$$\nabla^2 \vec{A} \equiv \left(\nabla^2 A_x\right)\hat{x} + \left(\nabla^2 A_y\right)\hat{y} + \left(\nabla^2 A_z\right)\hat{z}.$$

This is nothing more than a convenient extension of the meaning of ∇^2.

(2) Curl of gradient: $\vec{\nabla} \times \left(\vec{\nabla} V\right)$

The divergence $\vec{\nabla} \cdot \vec{A}$ is a *scalar*-all we can do is taking its gradient.

The curl of a gradient is always *zero*: $\vec{\nabla} \times \left(\vec{\nabla} V\right) = 0.$

(3) Gradient of divergence: $\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)$

The curl $\vec{\nabla} \times \vec{A}$ is a *vector*, so we can take its *divergence* and *curl*.

Notice that $\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right)$ is not the same as the Laplacian of a vector:

$$\nabla^2 \vec{A} = \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{A} \neq \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right).$$

(4) Divergence of curl: $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right)$

The divergence of a curl, like the curl of a gradient, is *always zero*:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0.$$

(5) Curl of curl: $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)$

As you can check from the definition of $\vec{\nabla}$:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}.$$

So curl-of-curl gives nothing new; the first term is just number (3) and the second is the Laplacian (of a vector).

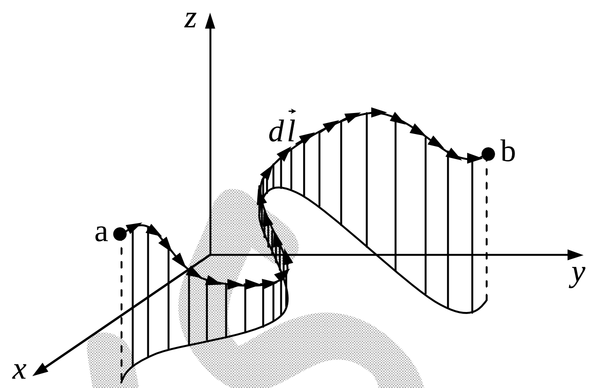
1.4 Integral Calculus

1.4.1 Line, Surface, and Volume Integrals

(a) Line Integrals

A line integral is an expression of the form

$$\int_a^b \vec{A}\cdot d\vec{l},$$



where $\vec{A}$ is a vector function, $d\vec{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path P from point a to point b. If the path in question forms a closed loop (that is, if $b = a$), put a circle on the integral sign:

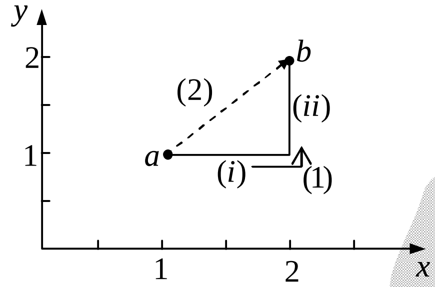
$$\oint \vec{A}\cdot d\vec{l}.$$

At each point on the path we take the dot product of $\vec{A}$ (evaluated at that point) with the displacement $d\vec{l}$ to the next point on the path. The most familiar example of a line integral is the work done by a force $\vec{F}$:

$$W = \int \vec{F}\cdot d\vec{l}$$

Ordinarily, the value of a line integral depends critically on the particular path taken from a to b, but there is an important special class of vector functions for which the line integral is independent of the path, and is determined entirely by the end points(A *force* that has this property is called ***conservative***.)

Example: Calculate the line integral of the function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point a = (1, 1, 0) to the point b = (2, 2, 0), along the paths (1) and (2) as shown in figure. What is $\oint \vec{A}\cdot d\vec{l}$ for the loop that goes from a to b along (1) and returns to a along (2)?



Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x},\quad y = 1,\quad \vec{A}\cdot d\vec{l} = y^2 dx = dx$, so $\int \vec{A}\cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y},\quad x = 2,\quad \vec{A}\cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy$, so $\int \vec{A}\cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$.

By path (1), then,

$$\int_a^b \vec{A}\cdot d\vec{l} = 1 + 10 = 11$$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y},\quad \vec{A}\cdot d\vec{l} = x^2 dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

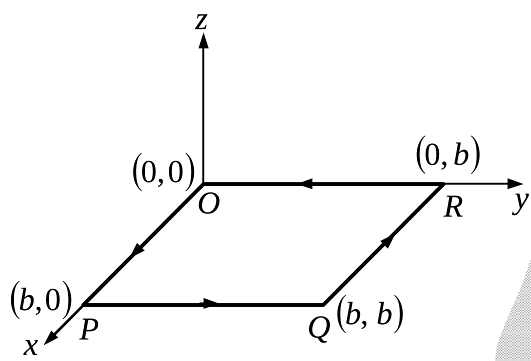
so

$$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 (3x^2 + 2x)dx = (x^3 + x^2)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then,

$$\oint \vec{A}\cdot d\vec{l} = 11 - 10 = 1$$

Example: Find the line integral of the vector $\vec{A} = \left(x^2 - y^2\right)\hat{x} + 2xy\hat{y}$ around a square of side 'b' which has a corner at the origin, one side on the x axis and the other side on the y axis.



Solution: In a Cartesian coordinate system $d\hat{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, $\vec{A} = \left(x^2 - y^2\right)\hat{x} + 2xy\hat{y}$

$$\oint_{OPQRO} \vec{A}\cdot d\vec{l} = \oint_{OPQRO}\left[\left(x^2 - y^2\right)dx + 2xydy\right]$$

Along OP, $y = 0,\ dy = 0 \Rightarrow \int_{OP} \vec{A}\cdot d\vec{l} = \int_{x=0}^{b} x^2 dx = \dfrac{b^3}{3}$

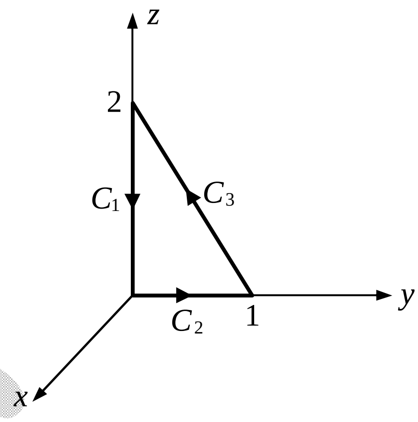
Along PQ, $x = b,\ dx = 0 \Rightarrow \int_{PQ} \vec{A}\cdot d\vec{l} = \int_{y=0}^{b} 2b\,y\,dy = b^3$

Along QR, $y = b,\ dy = 0 \Rightarrow \int_{QR} \vec{A}\cdot d\vec{l} = \int_{x=b}^{0} \left(x^2 - b^2\right)dx = \left(\dfrac{x^2}{3} - b^2 x\right)\Bigg|_{x=b}^{0} = \dfrac{2}{3}b^3$

Along RO, $x = 0,\ dx = 0 \Rightarrow \int_{RO} \vec{A}\cdot d\vec{l} = 0$

$$\oint \vec{A}\cdot d\vec{l} = \int_{OP} \vec{A}\cdot d\vec{l} + \int_{PQ} \vec{A}\cdot d\vec{l} + \int_{QR} \vec{A}\cdot d\vec{l} + \int_{RO} \vec{A}\cdot d\vec{l} = \frac{b^3}{3} + b^2 + \frac{2}{3}b^3 + 0 = 2b^3$$

Example: Compute the line integral $\vec{F} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ along the triangular path shown in figure.



Solution: Line Integral $\oint \vec{F}.d\vec{l} = \int_{C_1} \vec{F}.d\vec{l} + \int_{C_2} \vec{F}.d\vec{l} + \int_{C_3} \vec{F}.d\vec{l}$

On path C_1, $x = 0,\ y = 0,\ d\vec{l} = dz\hat{z}$

$$\int_{C_1} \vec{F}\cdot d\vec{l} = \int_{z=2}^{0} \left[6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}\right]\cdot dz\hat{z} = \int_{z=2}^{0} zdz = \left.\frac{z^2}{2}\right|_2^0 = -\frac{4}{2} = -2$$

On path C_2, $x = 0,\ z = 0,\ d\vec{l} = dy\hat{y} \Rightarrow \int_{C_2} \vec{F}\cdot d\vec{l} = \int_{y=0}^{1} yz^2 dy = 0$

On path C_3 the slope of line is-2 and intercept on z axis is $2 \Rightarrow z = -2y + 2 = 2(1-y)$ and the connecting points are (0, 1, 0) and (0, 0, 2)

On C_3, $x=0,\ dx = 0\quad d\vec{l} = dy\hat{y} + dz\hat{z}$

$$\int_{C_3} \vec{F}\cdot d\vec{l} = \int_{C_3} (yz^2)dy + (3y+z)dz = \int_{y=1}^{0} y[2(1-y)]^2 dy + \int_{z=0}^{2}\left[3\left(\frac{2-z}{2}\right) + z\right]dz$$

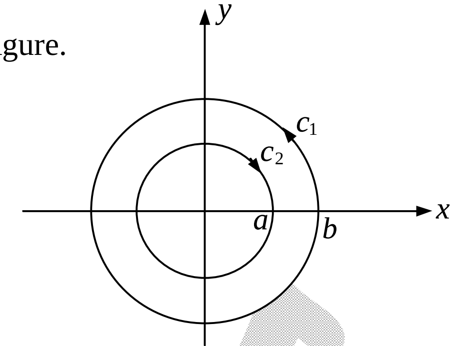
$$= \int_1^0 (4y + 4y^3 - 8y^2)dy + \int_{z=0}^{2}\left(3 - \frac{z}{2}\right)dz = 4\left.\frac{y^2}{2}\right|_1^0 + \left.\frac{4y^4}{4}\right|_1^0 - \left.\frac{8y^3}{3}\right|_1^0 + 3z\Big|_0^2 - \frac{1}{2}\left.\frac{z^2}{2}\right|_0^2$$

$$= -2 - 1 + \frac{8}{3} + 6 - 1 = \frac{14}{3}$$

$$\oint \vec{F}.d\vec{l} = \int_{C_1} \vec{F}.d\vec{l} + \int_{C_2} \vec{F}.d\vec{l} + \int_{C_3} \vec{F}.d\vec{l} = -2 + 0 + \frac{14}{3} = \frac{8}{3}$$

Example: Given $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$ in cylindrical coordinates. Find $\oint_{C_1} \vec{A}\cdot d\vec{l} + \oint_{C_2} \vec{A}\cdot d\vec{l}$

where c_1 and c_2 are contours shown in figure.



Solution: In cylindrical coordinate system $d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$, $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$

$$\vec{A}\cdot d\vec{l} = 2r\cos\phi dr + r^2 d\phi$$

In figure on curve c_1, ϕ varies from 0 to $2\pi, r = b$ and $dr = 0$

$$\oint_{c_1} \vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=b}}^{2\pi} r^2 d\phi = 2\pi b^2$$

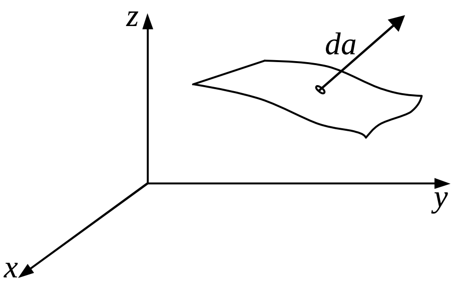
On curve $c_2, r = a, \phi$ varies from 0 to -2π, and $dr = 0 \Rightarrow \oint_{c_2} \vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=a}}^{-2\pi} r^2 d\phi = -2\pi a^2$

So, $\oint_{c_1} \vec{A}\cdot d\vec{l} + \oint_{c_2} \vec{A}\cdot d\vec{l} = 2\pi\left(b^2 - a^2\right)$

(b) Surface Integrals

A surface integral is an expression of the form

$$\int_S \vec{A}\cdot d\vec{a}$$



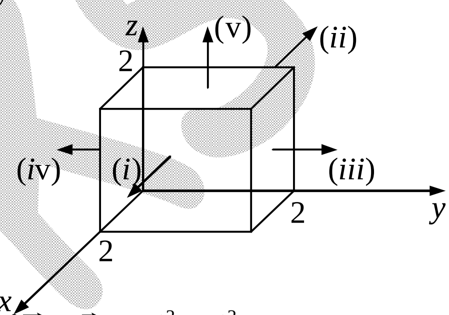
where $\vec{A}$ is again some vector function, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface(as shown in figure). There are, of course, two directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous. If the surface is closed then "outward" is positive, but for open surfaces it's arbitrary.

If $\vec{A}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{A}\cdot d\vec{a}$ represents the total mass per unit time passing through the surface-hence the alternative name, "flux."

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, and is determined entirely by the boundary line.

Example: Calculate the surface integral of $\vec{A} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) as shown in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.

Solution: Taking the sides one at a time:



(i) $x = 2,\ d\vec{a} = dydz\hat{x},\ \vec{A}\cdot d\vec{a} = 2xzdydz = 4zdydz,$ so $\int \vec{A}\cdot d\vec{a} = 4\int_0^2 dy\int_0^2 zdz = 16$.

(ii) $x = 0,\ d\vec{a} = -dydz\hat{x},\ \vec{A}\cdot d\vec{a} = -2xzdydz = 0,$ so $\int \vec{A}\cdot d\vec{a} = 0$.

(iii) $y = 2,\ d\vec{a} = dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = (x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 (x+2)dx\int_0^2 dz = 12$.

(iv) $y = 0,\ d\vec{a} = -dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = -(x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = -\int_0^2 (x+2)dx\int_0^2 dz = -12$.

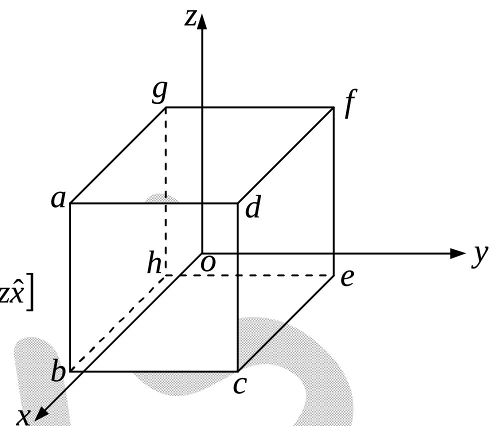
(v) $z = 2,\ d\vec{a} = dx\,dy\,\hat{z},\ \vec{A}\cdot d\vec{a} = y(z^2-3)dx\,dy = y\,dx\,dy,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 dx\int_0^2 ydy = 4$

Evidently the total flux is

$$\int_{\text{surface}} \vec{A}\cdot d\vec{a} = 16 + 0 + 12 - 12 + 4 = 20$$

Example: Given a vector $\vec{A} = (x^2 - y^2)\hat{x} + 2xy\hat{y} + (x^2 - xy)\hat{z}$. Evaluate $\oint_S \vec{A}\cdot d\vec{a}$ over the surface of the cube with the centre at the origin and length of side 'a'.

Solution: The surface integral is performed on all faces. The differential surface on the different faces are $\pm\, dy\,dz\hat{x}, \pm dx\,dz\hat{y},$ and $\pm\, dx\,dy\hat{z}$



Face $abcd$, $x = \frac{a}{2}$

$$\int_{abcd} \vec{A}\cdot d\vec{a} = \int_S \left[(x^2 - y^2)\hat{x} + 2xy\hat{y} + (x^2 - xy)\hat{z}\right]\cdot\left[dy\,dz\hat{x}\right]$$

$$= \int_{y=-a/2}^{+a/2}\int_{z=-a/2}^{a/2} (x^2 - y^2)\,dydz = \frac{a^4}{6}$$

Face $efgh$, $x = -\frac{a}{2} \Rightarrow \int_{efgh} \vec{A}\cdot d\vec{a} = \int_{efgh} \vec{A}\cdot(-dy\,dz\hat{x}) = \int_{y=-a/2}^{a/2}\int_{z=-a/2}^{a/2} -(x^2 - y^2)\,dy\;\;dz = -\frac{a^4}{6}$

Face $cdfe$, $y = \frac{a}{2} \Rightarrow \int_{cdfe} \vec{A}\cdot d\vec{a} = \int_S \vec{A}\cdot dx\,dz\hat{y} = \int_{x=-a/2}^{a/2}\int_{z=-a/2}^{a/2} 2xy\,dx\,dz = 0$

Face $aghb$, $y = -\frac{a}{2} \Rightarrow \int_{aghb} \vec{A}\cdot d\vec{a} = \int_{x=-a/2}^{a/2}\int_{z=-a/2}^{a/2} \vec{A}\cdot(-dx\,dz\hat{y}) = 0$

Similarly for the other two faces $adfg$ and $bceh$ we can find the surface integral with $d\hat{a} = \pm dx\,dy\hat{z}$, respectively. The addition of these two surface integrals will be zero.

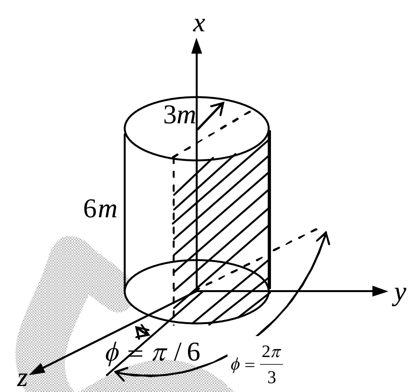
In the present case sum of all the surface integral

$$\oint_S \vec{A}.d\vec{a} = 0$$

Example: Use the cylindrical coordinate system to find the area of a curved surface on the right circular cylinder having radius = 3 m and height = 6 m and $30^0 \le \phi \le 120^0$.

Solution: From figure, surface area is required for a cylinder when r = 3m, z = 0 to 6m,

$$30^0 \le \phi \le 120^0 \text{ or } \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$$

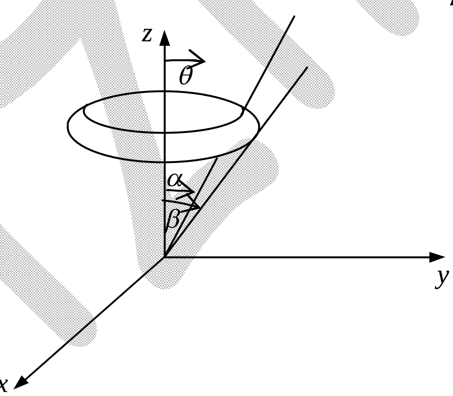


In cylindrical coordinate system, the elemental surface area as scalar is $d\vec{a} = r\,d\phi\,dz\hat{r}$

Taking the magnitude only

$$A = \int_S da = \int_{\phi=\pi/6}^{2\pi/3} \int_{z=0}^{6} r\,d\phi\,dz = 3\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)6 = 9\pi\,m^2$$

Example: Use spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius 'a'. Calculate the area when $\alpha = 0$ and $\beta = \pi$.



Solution: Sphere has radius 'a' and θ varies between α and β.

For fixed radius the elemental surface is $da = (r\sin\theta\,d\phi)(r\,d\theta) = r^2\sin\theta\,d\theta\,d\phi$

Area $A = \int_{\theta=\alpha}^{\beta}\int_{\phi=0}^{2\pi} r^2\sin\theta\,d\theta\,d\phi = 2\pi a^2\int_{\theta=\alpha}^{\beta}\sin\theta\,d\theta = 2\pi a^2(\cos\alpha - \cos\beta)$

For $\alpha = 0, \beta = \pi$, Area $= 2\pi a^2(1+1) = 4\pi a^2$, is surface area of the sphere.

(c) Volume Integrals

A volume integral is an expression of the form

$$\int_V T\, d\tau,$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\, dy\, dz.$$

For example, if T is the density of a substance (which might vary from point to point) then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of vector functions:

$$\int \vec{A}\, d\tau = \int \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) d\tau = \hat{x}\int A_x d\tau + \hat{y}\int A_y d\tau + \hat{z}\int A_z d\tau\,;$$

because the unit vectors are constants, they come outside the integral.

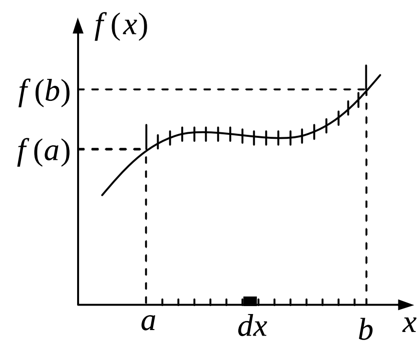
1.4.2 The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The **fundamental theorem of calculus** states:

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a) \quad \text{or} \quad \int_a^b F(x)dx = f(b) - f(a)$$

where $df/dx = F(x)$.

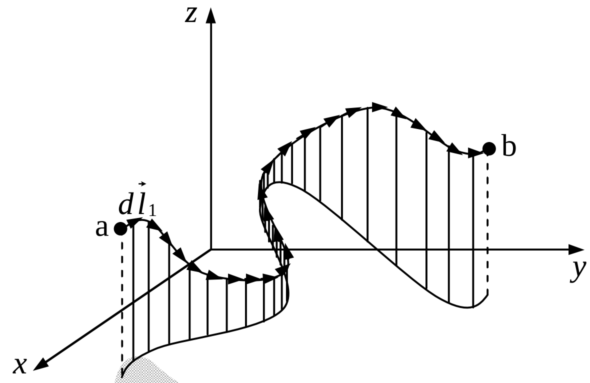
Geometrical Interpretation



According to equation $df = (df/dx)\, dx$ is the infinitesimal change in f when one goes from (x) to $(x + dx)$. The fundamental theorem says that if you chop the interval from a to b into many tiny pieces, dx, and add up the increments df from each little piece, the result is equal to the total change in f is $f(b) - f(a)$.

In other words, there are two ways to determine the total change in the function: either subtract the values at the ends or go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

1.4.3 The Fundamental Theorem for Gradients



Suppose we have a scalar function of three variables $V(x, y, z)$. Starting at point a, we moves a small distance $d\vec{l}_1$. Then

$$dV = \left(\vec{\nabla}V\right)\cdot d\vec{l}_1 .$$

Now we move a little further, by an additional small displacement $d\vec{l}_2$; the incremental change in V will be $\left(\vec{\nabla}V\right)\cdot d\vec{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point b. At each step we compute the gradient of V (at that point) and dot it into the displacement $d\vec{l}$ ...this gives us the change in V. Evidently the total change in V in going from a to b along the path selected is

$$\int_{a\,P}^{b}\left(\vec{\nabla}V\right)\cdot d\vec{l} = V(b) - V(a).$$

This is called the fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b).

Geometrical Interpretation

Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem).

Corollary 1: $\int_a^b\left(\vec{\nabla}V\right)\cdot d\vec{l}$ is independent of path taken from a to b.

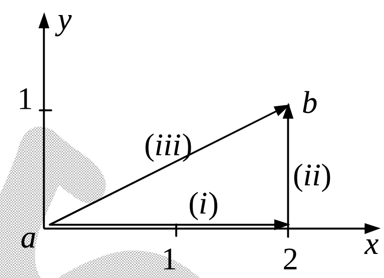
Corollary 2: $\oint\left(\vec{\nabla}V\right)\cdot d\vec{l} = 0$, since the beginning and end points are identical, and hence

$V(b) - V(a) = 0$.

Example: Let $V = xy^2$, and take point a to be the origin (0, 0, 0) and b the point (2, 1, 0). Check the fundamental theorem for gradients.

Solution: Although the integral is independent of path, we must pick a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii). As always,

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z},\quad \vec{\nabla}V = y^2\hat{x} + 2xy\,\hat{y}$$



(i) $y = 0;\ d\vec{l} = dx\,\hat{x},\ \vec{\nabla}V\cdot d\vec{l} = y^2 dx = 0,$ so $\int_i \vec{\nabla}V\cdot d\vec{l} = 0$

(ii) $x = 2;\ d\vec{l} = dy\,\hat{y},\ \vec{\nabla}V.d\vec{l} = 2xydy = 4y\,dy,$ so

$$\int_{ii}\vec{\nabla}V\cdot d\vec{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$$

Evidently the total line integral is 2.

This consistent with the fundamental theorem: T(b) – T(a) =2 – 0 = 2.

Calculate the same integral along path (iii) (the straight line from a to b):

(iii) $y = \frac{1}{2}x, dy = \frac{1}{2}dx, \vec{\nabla}V\cdot d\vec{l} = y^2 dx + 2xy\,dy = \frac{3}{4}x^2 dx,$ so

$\int_{iii}\vec{\nabla}V\cdot d\vec{l} = \int_0^2 \frac{3}{4}x^2\,dx = \frac{1}{4}x^3\Big|_0^2 = 2$. Thus the integral is independent of path.

1.4.4 The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$$

This theorem has at least three special names: ***Gauss's theorem, Green's theorem***, or, simply, the ***divergence theorem***. Like the other "fundamental theorems," it says that the integral of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume). Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

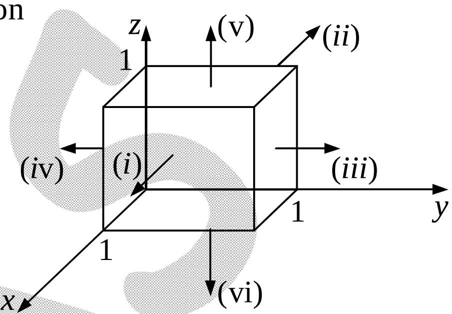
Geometrical Interpretation

If $\vec{A}$ represents the flow of an incompressible fluid, then "the flux of $\vec{A}$ (the right side of equation) is the total amount of fluid passing out through the surface, per unit time and the left side of equation shows an equal amount of liquid will be forced out through the boundaries of the region.

Example: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + \left(2yz\right)\hat{z}$$

and the unit cube situated at the origin.



Solution: In this case

$$\vec{\nabla}\cdot\vec{A} = 2(x+y),$$

and $$\int_V 2(x+y)d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)dx = \frac{1}{2}+y,\ \int_0^1\left(\frac{1}{2}+y\right)dy = 1,\ \int_0^1 1dz = 1.$$

Evidently, $$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3}$

(ii) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\frac{1}{3}$

(iii) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 \left(2x+z^2\right)dx\,dz = \frac{4}{3}$

(iv) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$

(v) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1$

(vi) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 0\,dx\,dy = 0$

So the total flux is:

$$\oint_S \vec{A}\cdot d\vec{a} = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2.$$

Example: A vector field $\vec{A} = \left(\frac{5r^2}{4}\right)\hat{r}$ is given in spherical coordinates. Evaluate both sides of Divergence Theorem for the volume enclosed between

(i) $r = 1$ and $r = 2$, and

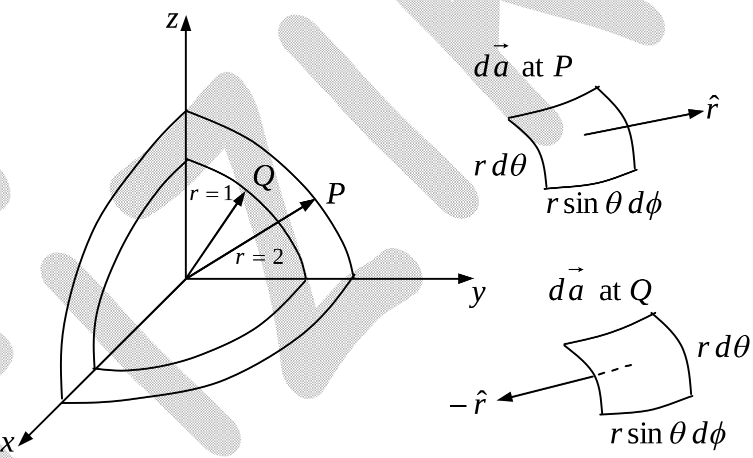
(ii) $\theta = 0$ to $\theta = \frac{\pi}{4}$ and $r = 4$.

Solution: Divergence theorem states that $\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$

Since $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = \frac{5r^2}{4},\quad A_\theta = 0,\quad A_\phi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{5}{4}r^2\right) = 5r$$

(i)



$$L.H.S = \int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_v (5r)r^2\sin\theta\,dr\,d\theta\,d\phi = \int_{r=1}^{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 5r^3\sin\theta\,dr\,d\theta\,d\phi = 75\pi$$

$$\text{R.H.S} \oint_S \vec{A}\cdot d\vec{a} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(r^2\sin\theta\,d\theta\,d\phi\hat{r}\right) + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(-r^2\sin\theta\,d\theta\,d\phi\hat{r}\right)$$

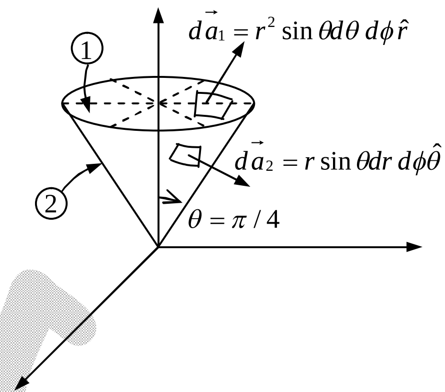
$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(2)^4\sin\theta\,d\theta\,d\phi - \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(1)^4\sin\theta\,d\theta\,d\phi = 75\pi$$

So L.H.S. = R.H.S. = 75π

Divergence theorem proved.

(ii) L.H.S. of Divergence Theorem

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_{r=0}^{4}\int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}(5r)\cdot r^2\sin\theta\, dr\, d\theta\, d\phi = 588.91$$



R.H.S. of Divergence Theorem

$$\oint_S \vec{A}\cdot d\vec{a} = \int_{S_1}\vec{A}\cdot d\vec{a} + \int_{S_2}\vec{A}\cdot d\vec{a}$$

$$= \int_{S_1}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\hat{r}\right) + \int_{S_2}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r\sin\theta\, dr\, d\phi\,\hat{\theta}\right)$$

$$= \int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}\frac{5}{4}(4)^4\sin\theta\, d\theta\, d\phi + 0 \;= 588.91$$

L.H.S. = R.H.S. = 588.91. Divergence theorem proved.

1.4.5 The Fundamental Theorem for Curls

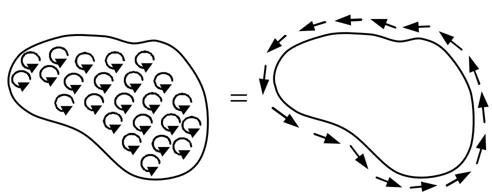
The fundamental theorem for curls, which goes by the special name of ***Stokes' theorem***, states that

$$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint_P \vec{A}\cdot d\vec{l}$$

As always, the integral of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (here, the perimeter of the patch). As in the case of the divergence theorem, the boundary term is itself an integral-specifically, a closed line integral.

Geometrical Interpretation:

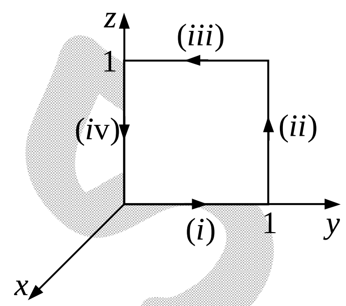
The integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (as shown in figure).



Corollary 1: $\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of equation vanishes.

Example: Suppose $\vec{A} = (2xz+3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



Solution: Here

$\vec{\nabla}\times\vec{A} = (4z^2-2x)\hat{x} + 2z\,\hat{z}$ and $d\vec{a} = dy\,dz\,\hat{x}$

(In saying that $d\vec{a}$ points in the x direction, we are chosen to a counterclockwise line integral. We could as well write $d\vec{a} = -dy\,dz\,\hat{x}$, but then we have to go clockwise.) Since $x = 0$ for this surface,

$$\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_0^1\int_0^1 4z^2\,dy\,dz = \frac{4}{3}$$

Now, what about the line integral? We must break this up into four segments:

(i) $x=0,\quad z=0,\quad \vec{A}\cdot d\vec{l} = 3y^2dy,\qquad \int \vec{A}\cdot d\vec{l} = \int_0^1 3y^2dy = 1,$

(ii) $x=0,\quad y=1,\quad \vec{A}\cdot d\vec{l} = 4z^2dz,\ \int \vec{A}\cdot d\vec{l} = \int_0^1 4z^2dz = \frac{4}{3},$

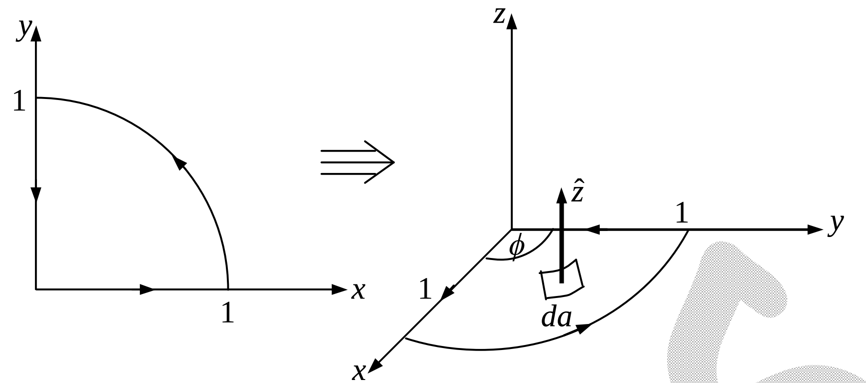
(iii) $x=0,\quad z=1,\quad \vec{A}\cdot d\vec{l} = 3y^2dy,\qquad \int \vec{A}\cdot d\vec{l} = \int_1^0 3y^2dy = -1,$

(iv) $x=0,\quad y=0,\quad \vec{A}\cdot d\vec{l} = 0,\qquad \int \vec{A}\cdot d\vec{l} = \int_1^0 0dz = 0,$

So

$$\oint \vec{A}\cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$

Example: Given $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$ in cylindrical coordinates. For the contour shown in figure, verify the Stokes' Theorem.



Solution: Stokes' Theorem $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$

In cylindrical coordinates, $\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$

$$A_r = 2r\cos\phi,\quad A_\phi = r\,,\, A_z = 0$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{-\partial}{\partial z}\left(r^2\right)\right]\hat{r} + \left[\frac{\partial}{\partial z}\left(2r\cos\phi\right)\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^2\right)\frac{\partial}{\partial \phi}\left(2r\cos\phi\right)\right]\hat{z} = \left(2+2\sin\phi\right)\hat{z}$$

L.H.S.= $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\left(2+2\sin\phi\right)\hat{z}\cdot\left(r\,dr\,d\phi\hat{z}\right) = \frac{\pi}{2}+1$

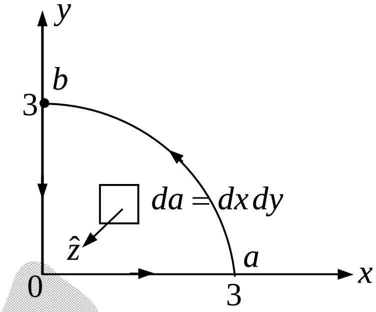
R.H.S. $= \oint \vec{A}\cdot d\vec{l} = \int_{r=0,1}\vec{A}\cdot d\vec{l} + \int_{\phi=0,\pi/2}\vec{A}\cdot d\vec{l} + \int_{r=1,0}\vec{A}\cdot d\vec{l}$

$$\vec{A}\cdot d\vec{l} = \left(2r\cos\phi\hat{r} + r\hat{\phi}\right)\cdot\left(dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}\right) = 2r\cos\phi\,dr + r^2 d\phi$$

$$\oint \vec{A}\cdot d\vec{l} = \int_{\substack{r=0\\ at\,\phi=0}}^{1} 2r\cos\phi\,dr + \int_{\substack{\phi=0\\ at\,r=1}}^{\pi/2} r^2 d\phi + \int_{\substack{r=1\\ at\,\phi=\pi/2}}^{0} 2r\cos\phi\,dr \quad = 1+\frac{\pi}{2}+0 = 1+\frac{\pi}{2}$$

L.H.S. = R.H.S. $= 1+\frac{\pi}{2}$

Example: Given a vector field $\vec{A} = xy\hat{x} - 2x\hat{y}$. Verify stokes,' theorem over the path shown in figure.



Solution: Stokes' theorem $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -(2+x)\hat{z}$$

L.H.S. $= \int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}} \left(\vec{\nabla}\times\vec{A}\right)\cdot\left(dx\,dy\hat{z}\right)$, since $r^2 = x^2+y^2$ or $x = \sqrt{9-y^2}$

$$= \int_0^3\left[\int_0^{\sqrt{9-y^2}} -(2+x)dx\,dy\right] = \int_0^3\left[\left|-2x-\frac{x^2}{2}\right|_0^{\sqrt{9-y^2}} dy\right] = \int_0^3 -\left[+2\sqrt{9-y^2}+\left(\frac{9-y^2}{2}\right)\right]dy$$

$$= -\left[y\sqrt{9-y^2}+9\sin^{-1}\frac{y}{3}+\frac{9}{2}y-\frac{y^3}{6}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

R.H.S. $= \oint \vec{A}\cdot d\vec{l} = \int_{0,a}\vec{A}\cdot d\vec{l} + \int_{a,b}\vec{A}\cdot d\vec{l} + \int_{b,0}\vec{A}\cdot d\vec{l}$

On a, $y=0$; $\int \vec{A}\cdot d\vec{l} = \int -2x\,dy = 0$

On ab; $\int \vec{A}\cdot d\vec{l} = \int\left(xy\,dx - 2x\,dy\right) = \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3\sqrt{9-y^2}\,dy$

(Equation of quarter circle $x^2+y^2=9;\ \ 0\le x, y\le 3$)

$$\int \vec{A}\cdot d\vec{l} = -\frac{1}{3}\left(9-x^2\right)^{3/2}\Big|_3^0 - \left[y\sqrt{9-y^2}+9\sin^{-1}\frac{y}{3}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

On $b0$, x = 0; $\int \vec{A}\cdot d\vec{l} = 0$

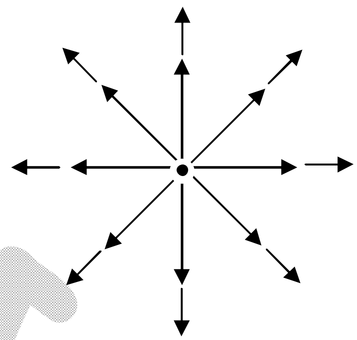
$$\Rightarrow \oint \vec{A}\cdot d\vec{l} = -9\left(1+\frac{\pi}{2}\right)$$

L.H.S. = R.H.S $= -9\left(1+\frac{\pi}{2}\right)$

1.5 The Dirac Delta Function

1.5.1 The Divergence of $\hat{r}/r^2$

Consider the vector function



$$\vec{A} = \frac{1}{r^2}\hat{r}$$

At every location, $\vec{A}$ is directed radially outward. When we calculate the divergence we get precisely zero:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

The plot thickens if you apply the divergence theorem to this function. Suppose we integrate over a sphere of radius *R*, centered at the origin; the surface integral is

$$\oint \vec{A}\cdot d\vec{a} = \int\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right) = \left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

But the volume integral, $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau$, is zero. Does this mean that the divergence theorem is false?

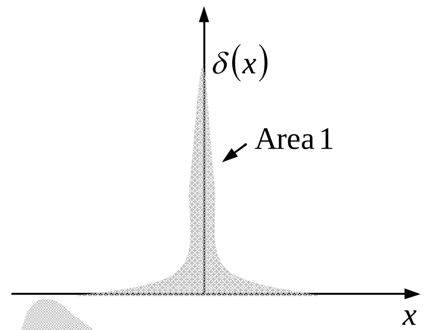
The source of the problem is the point $r = 0$, where $\vec{A}$ blows up. It is quite true that $\vec{\nabla}\cdot\vec{A} = 0$ everywhere except the origin, but right at the origin the situation is more complicated.

Notice that the surface integral is independent of *R*; if the divergence theorem is right (and it is), we should get $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 4\pi$ for any sphere centered at the origin, no matter how small. Evidently the entire contribution must be coming from the point $r = 0$!

Thus, $\vec{\nabla}\cdot\vec{A}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π. No ordinary function behaves like that. (On the other hand, a physical example does come to mind: the density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite namely, the mass of the particle.) What we have stumbled on is a mathematical object known to physicists as the ***Dirac delta function***.

1.5.2 The One- Dimensional Dirac Delta Function

The one dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1 (as shown in figure).



That is to say:

$$\delta(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ \infty, & \text{if} \quad x = 0 \end{cases}$$

and $$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

If $f(x)$ is some "ordinary" function then the product $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows that

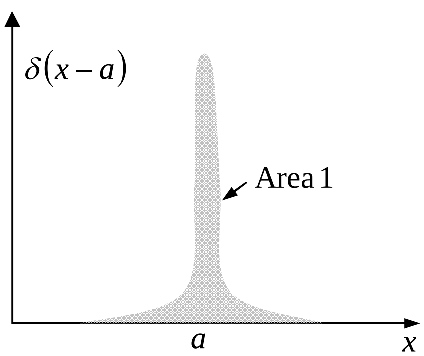
$$f(x)\delta(x) = f(0)\delta(x)$$

Of course, we can shift the spike from $x = 0$ to some other point, $x = a$:

$$\delta(x-a) = \begin{cases} 0, & \text{if} \quad x \neq a \\ \infty, & \text{if} \quad x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(\text{x-a})\,dx = 1$$

also $$f(x)\delta(x-a) = f(a)\delta(x-a)$$

and $$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$



Example: Show that

$$\delta(kx)=\frac{1}{|k|}\delta(x),$$

where k is any (nonzero constant). (In particular $\delta(-x)=\delta(x)$.)

Solution: For an arbitrary test function $f(x)$, consider the integral $\int_{-\infty}^{\infty} f(x)\delta(kx)dx$.

Changing variables, we let $y=kx$, so that $x=y/k\,x$, and $dx=\frac{1}{k}dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is negative, then $x=\infty$ implies $y=-\infty$, and vice versa, so the order of limit is reversed. Restoring the "proper" order costs a minus sign. Thus

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx=\pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k}=\pm\frac{1}{k}f(0)=\frac{1}{|k|}f(0)$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k, as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx=\int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx\Rightarrow\delta(kx)=(1/|k|)\delta(x)$$

Example: Evaluate the integral $I=\int_0^1 x^3\delta(x-2)\,dx$

Solution: Answer would be 0, because the spike would then be outside the domain of integration.

Example: Evaluate the integral $I=\int_0^3 x^3\delta(x-2)\,dx$

Solution: The delta function picks out the value of x^3 at the point $x=2$ so the integral is $2^3=8$.

Example: Show that

$$x\frac{d}{dx}(\delta(x)) = -\delta(x) \qquad \text{and} \qquad \frac{d^n}{dx^n}(\delta(x)) = (-1)^n n!\frac{\delta(x)}{x^n}$$

Solution: $\int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}(\delta(x))\right]dx = xf(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}(xf(x))\delta(x)\,dx$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}(\delta(x))\right]dx = -\int_{-\infty}^{\infty}\frac{d}{dx}(xf(x))\delta(x)\,dx \qquad \because xf(x)\delta(x)\Big|_{-\infty}^{\infty} = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}(\delta(x))\right]dx = -\int_{-\infty}^{\infty}\left(x\frac{df}{dx}+f\right)\delta(x)\,dx = -f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)\,dx$$

$$\Rightarrow x\frac{d}{dx}(\delta(x)) = -\delta(x) \qquad \Rightarrow \frac{d}{dx}(\delta(x)) = -\frac{\delta(x)}{x}$$

$$\Rightarrow \frac{d^2}{dx^2}(\delta(x)) = \frac{d}{dx}\left(\frac{d}{dx}\delta(x)\right) = \frac{d}{dx}\left(-\frac{\delta(x)}{x}\right) = -\left(\delta(x)\frac{-1}{x^2} + \frac{1}{x}\frac{d}{dx}\delta(x)\right)$$

$$\Rightarrow \frac{d^2}{dx^2}(\delta(x)) = -\left(\delta(x)\frac{-1}{x^2} + \frac{1}{x}\left(\frac{-\delta(x)}{x}\right)\right) = 2\frac{\delta(x)}{x^2}$$

Similarly $\frac{d^3}{dx^3}(\delta(x)) = -6\frac{\delta(x)}{x^3}$. Thus $\frac{d^n}{dx^n}(\delta(x)) = (-1)^n n!\frac{\delta(x)}{x^n}$

Example: Let $\theta(x)$ be the step function:

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

Show that $\frac{d\theta}{dx} = \delta(x)$.

Solution:

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta(x)\,dx = f(\infty) - \int_{0}^{\infty}\frac{df}{dx}dx = f(\infty) - \left[f(\infty) - f(0)\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx \Rightarrow \frac{d\theta}{dx} = \delta(x)$$

1.5.3 The Three-Dimensional Delta Function

It is an easy matter to generalize the delta function to three dimensions:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

(As always, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector, extending from the origin to the point (x, y, z)). This three-dimensional delta function is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1

$$\int_{all\ space} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dx\,dy\,dz = 1$$

and

$$\int_{all\ space} f(\vec{r})\delta^3(\vec{r}-\vec{a})d\tau = f(\vec{a})$$

Since the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, and yet its integral over any volume containing the origin is a constant (4π). These are precisely the defining conditions for the Dirac delta function; evidently

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

Example: Evaluate the integral $J = \int_v (r+1)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau$ where v is a sphere of radius R centered at the origin.

Solution: $J = \int_v (r+1)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_v (r+1)4\pi\delta^3(r)\,d\tau = 4\pi(0+1) = 4\pi$

1.6 The Theory of Vector Fields

If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a scalar potential (V): $\vec{\nabla}\times\vec{F}=0 \Leftrightarrow \vec{F}=-\vec{\nabla}V$

(The minus sign is purely conventional.)

Theorem 1: Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, $\vec{F}$ satisfies one if and only if it satisfies all the others):

(a) $\vec{\nabla}\times\vec{F}=0$ everywhere.

(b) $\int_a^b \vec{F}\cdot d\vec{l}$ is independent of path, for any given end points.

(c) $\oint \vec{F}\cdot d\vec{l}=0$ for any closed loop.

(d) $\vec{F}$ is the gradient of some scalar, $\vec{F}=-\vec{\nabla}V$.

The scalar potential is not unique-any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of a vector potential $(\vec{A})$:

$$\vec{\nabla}\cdot\vec{F}=0 \Leftrightarrow \vec{F}=\vec{\nabla}\times\vec{A}$$

That's the main conclusion of the following theorem:

Theorem 2: Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

(a) $\vec{\nabla}\cdot\vec{F}=0$ everywhere.

(b) $\int \vec{F}\cdot d\vec{a}$ is independent of surface, for any given boundary line.

(c) $\oint \vec{F}\cdot d\vec{a}=0$ for any closed surface.

(d) $\vec{F}$ is the curl of some vector, $\vec{F}=\vec{\nabla}\times\vec{A}$.

The vector potential is not unique-the gradient of any scalar function can be added to $\vec{A}$ without affecting the curl, since the curl of a gradient is zero.

MCQ (Multiple Choice Questions)

Q1. Let $\vec{a}$ and $\vec{b}$ be two distinct three dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\frac{(\vec{a}\cdot\vec{b})\vec{b}}{b^2}$ (d) $\frac{(\vec{b}\cdot\vec{a})\vec{a}}{a^2}$

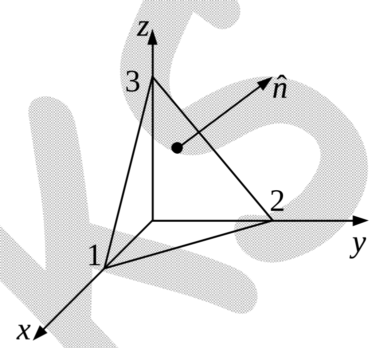
Q2. The components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure given below is:

(a) $\hat{n}=\frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$

(b) $\hat{n}=\frac{3\hat{x}+6\hat{y}+2\hat{z}}{7}$

(c) $\hat{n}=\frac{6\hat{x}+2\hat{y}+3\hat{z}}{7}$

(d) $\hat{n}=\frac{2\hat{x}+3\hat{y}+6\hat{z}}{7}$



Q3. The equation of the plane that is tangent to the surface $xyz=8$ at the point $(1,2,4)$ is

(a) $x+2y+4z=12$ (b) $4x+2y+z=12$

(c) $x+4y+2=0$ (d) $x+y+z=7$

Q4. A vector perpendicular to any vector that lies on the plane defined by $x+y+z=5$, is

(a) $\hat{i}+\hat{j}$ (b) $\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}+\hat{k}$ (d) $2\hat{i}+3\hat{j}+5\hat{k}$

Q5. The unit normal vector of the point $\left[\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

(a) $\frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$ (b) $\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(c) $\frac{b\hat{i}+c\hat{j}+a\hat{k}}{\sqrt{a^2+b^2+c^2}}$ (d) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Q6. The equation of a surface of revolution is $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$. The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is

(a) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{\sqrt{10}}\hat{k}$ (b) $\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$

(c) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (d) $\sqrt{\frac{3}{10}}\hat{i}+\frac{2}{\sqrt{10}}\hat{k}$

Q7. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r=|\vec{r}|$. Then

(a) $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}\times\vec{r}=\vec{r}/r$ (b) $\vec{\nabla}\cdot\vec{r}=0$ and $\nabla^2 r=0$

(c) $\vec{\nabla}\cdot\vec{r}=3$ and $\nabla^2\vec{r}=\vec{r}/r^2$ (d) $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}\times\vec{r}=0$

Q8. For vector function $\vec{A}=10\hat{r}+5\sin\theta\hat{\theta}$ (in spherical polar coordinate) the value of $\vec{\nabla}.\vec{A}$ is:

(a) $(1+\sin\theta)(10/r)$ (b) $(1+\cos\theta)(10/r)$

(c) $(2+\sin\theta)(10/r)$ (d) $(2+\cos\theta)(10/r)$

Q9. For vector function $\vec{F}=\left(\frac{150}{r^2}\right)\hat{r}+10\hat{\phi}$ (in cylindrical coordinate) then $\vec{\nabla}.\vec{F}$ is

(a) $\frac{-150}{r^3}$ (b) $\frac{150}{r^3}$ (c) $\frac{-150}{r^2}$ (d) $\frac{150}{r^2}$

Q10. For vector function $\vec{A}=\left(e^{-r}/r\right)\hat{\theta}$ (in spherical polar coordinate) then $\vec{\nabla}\times\vec{A}$ is

(a) $\frac{e^{-r}}{r}\hat{\phi}$ (b) $-\frac{e^{-r}}{r}\hat{\phi}$ (c) $\frac{e^{-r}}{r}\hat{\theta}$ (d) $-\frac{e^{-r}}{r}\hat{\theta}$

Q11. A vector $\vec{A}=k\hat{\phi}$ is given (where k is a constant), in cylindrical coordinates. Then the value of $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$ is:

(a) $\frac{k}{r}\hat{z}$ (b) $\frac{k}{r}\hat{\phi}$ (c) $\frac{k}{r^2}\hat{z}$ (d) $\frac{k}{r^2}\hat{\phi}$

Q12. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F}=0$ (b) $\vec{\nabla}\cdot\vec{F}=0$ (c) $\vec{\nabla}V=0$ (d) $\nabla^2 V=0$

Q13. If $\vec{F}$ is a constant vector and $\vec{r}$ is the position vector then $\vec{\nabla}(\vec{F}\cdot\vec{r})$ would be

(a) $(\vec{\nabla}\cdot\vec{r})\vec{F}$ (b) $\vec{F}$ (c) $(\vec{\nabla}\cdot\vec{F})\vec{r}$ (d) $|\vec{r}|\vec{F}$

Q14. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}(\vec{A}\cdot(\vec{B}\times\vec{r}))$ is

(a) $\vec{A}\cdot\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Q15. If $\vec{A}=\hat{i}yz+\hat{j}xz+\hat{k}xy$, then the integral $\oint_C \vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x=0, x=a$ and $y=0,\ y=b$) is

(a) $\frac{1}{2}(a^3+b^3)$ (b) $\pi(ab^2+a^2b)$ (c) $\pi(a^3+b^3)$ (d) 0

Q16. If the surface integral of the field $\vec{A}(x,y,z)=2kx\hat{i}+ly\hat{j}-3mz\hat{k}$ over the closed surface of an arbitrary unit sphere is to be zero, then the relationship between k, l and m is

(a) $6k+l-6m=0$ (b) $2k+l-3m=0$

(c) $3k+6l-2m=0$ (d) $2/k+1/l-3/m=0$

Q17. For the vector field $\vec{A}=xz^2\hat{i}-yz^2\hat{j}+z(x^2-y^2)\hat{k}$, the volume integral of the divergence of $\vec{A}$ out of the region defined by $-a\le x\le a$, $-b\le y\le b$ and $0\le z\le c$ is:

(a) $\frac{4}{3}abc\left[a^2-b^2\right]$ (b) $\frac{2}{3}abc\left[a^2-b^2\right]$

(c) $\frac{1}{3}abc\left[a^2-b^2\right]$ (d) $abc\left[a^2-b^2\right]$

Q18. Consider a constant vector field $\vec{v} = v_0\hat{k}$. If $\vec{v} = \vec{\nabla}\times\vec{u}$ then one of the many possible vectors $\vec{u}$ is

(a) $v_0 x\hat{i}$ (b) $v_0 x\hat{j}$ (c) $v_0\hat{i}$ (d) $v_0\hat{j}$

Q19. Consider a vector field $\vec{v} = v_0\hat{k}$ and $\vec{u} = v_0 x\hat{j}$ where $\vec{v} = \vec{\nabla}\times\vec{u}$. Then the flux associated with the field $\vec{v}$ through the curved hemispherical surface defined by $x^2 + y^2 + z^2 = r^2;\ z > 0$ is

(a) 0 (b) $\pi v_0 r^2$ (c) $2\pi v_0 r^2$ (d) $3\pi v_0 r^2$

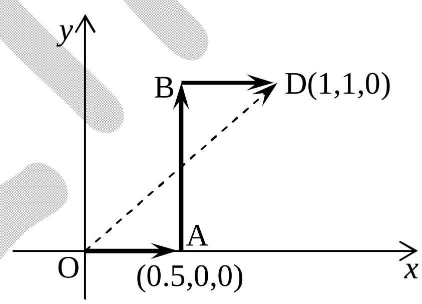
Q20. Consider a vector force $\vec{F}(x,y) = k\left[\left(x^2 + y^2\right)\hat{i} + 2xy\,\hat{j}\right]$. Here $1Fm^{-2}$. The work done by this force in moving a particle from the origin $O(0,0,0)$ to the point $D(1,1,0)$ on the $z=0$ plane along the path $OABD$ as shown in the figure is: (where the coordinates are measured in meters)

(a) $\frac{1}{3}$

(b) $\frac{2}{3}$

(c) $\frac{4}{3}$

(d) 0



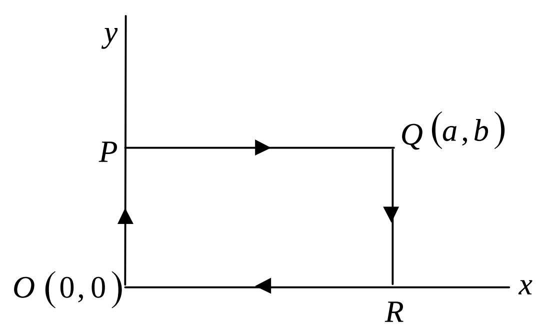
Q21. Consider force field $\vec{F}(x,y) = \left(x^2 - y^2\right)\hat{i} + 2xy\,\hat{j}$. Then the work done when an object moves from $O \to P \to Q \to R \to O$ along the rectangular path as shown in figure is:

(a) $-2ab^2$

(b) $+2ab^2$

(c) $-ab^2$

(d) $+ab^2$



Q22. Which of the following is correct expression for $\delta(kx)$, where k is any (nonzero constant) (In particular $\delta(-x)=\delta(x)$).

(a) $\delta(kx)=\frac{1}{|k|}\delta(x)$ (b) $\delta(kx)=|k|\delta(x)$

(c) $\delta(kx)=-\frac{1}{|k|}\delta(x)$ (d) $\delta(kx)=-|k|\delta(x)$

Q. 23. Evaluate the integral $\int_0^1 x^3\delta(x-2)\,dx$

(a) 0 (b) 4 (c) 8 (d) 12

Q24. Evaluate the integral $J=\int_v (r^2+2)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau$ where v is a sphere of radius R centered at the origin.

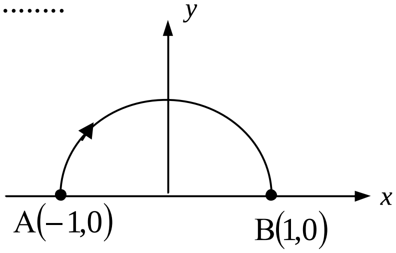
(a) 2π (b) 4π (c) 6π (d) 8π

Q25. Evaluate the integral $\int_0^3 x^3\delta(x-2)\,dx$

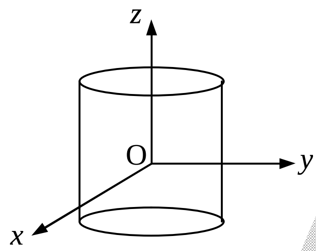
(a) 0 (b) 4 (c) 8 (d) 12

NAT (Numerical Answer Type)

Q26. The line integral $\int_A^B \vec{F}\cdot d\vec{l}$, where $\vec{F}=\frac{x}{\sqrt{x^2+y^2}}\hat{x}+\frac{y}{\sqrt{x^2+y^2}}\hat{y}$, along the semi-circular path as shown in the figure below is................



Q27. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,ds$ over the closed surface of the cylinder is $\alpha\pi a^2 h$. Then the value of α is.......



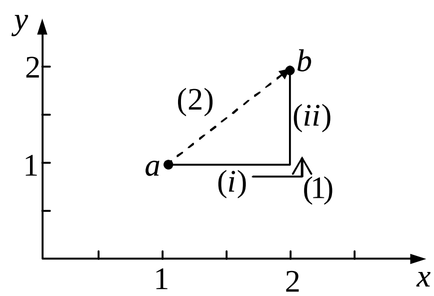
Q28. For vector function $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$ the value of $\vec{\nabla}.\vec{A}$ is........

Q29. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60° if $b = \alpha a$. Then the value of α is......

Q30. If S is the closed surface enclosing a volume V and $\hat{n}$ is the unit normal vector to the surface and $\vec{r}$ is the position vector, then the value of the following integral $\int\int_S \vec{r}.\hat{n}dS$ is αV. Then the value of α is ....

MSQ (Multiple Select Questions)

Q31. A vector function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ is given and two specified paths from a to b are shown in the figure given below. Coordinates of point a is $(1,1,0)$ and that of point b is $(2,2,0)$. Then which of the following statements are



(a) $\int_a^b \vec{A}\cdot d\vec{l} = 11$ along path 1.

(b) $\int_a^b \vec{A}\cdot d\vec{l} = 10$ along path 2.

(c) $\oint \vec{A}\cdot d\vec{l} = -1$ for the loop that goes from a to b along (1) and returns to a along (2).

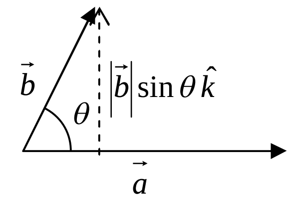
(d) $\oint \vec{A}\cdot d\vec{l} = 1$ for the loop that goes from a to b along (1) and returns to a along (2).

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (a)

Solution: $\vec{a}\times\vec{b}=ab\sin\theta\,\hat{n}$ where $\hat{n}$ is perpendicular to plane containing $\vec{a}$ and $\vec{b}$ and pointing upwards.



$$\vec{a}\times\left(\vec{a}\times\vec{b}\right)=ab\sin\theta\left(\vec{a}\times\hat{n}\right)=-a^2b\sin\theta\,\hat{k}$$

$$b\sin\theta\,\hat{k}=\frac{-\vec{a}\times\left(\vec{a}\times\vec{b}\right)}{a^2}\Rightarrow b\sin\theta\,\hat{k}=\frac{\vec{a}\times\left(\vec{b}\times\vec{a}\right)}{a^2}.$$

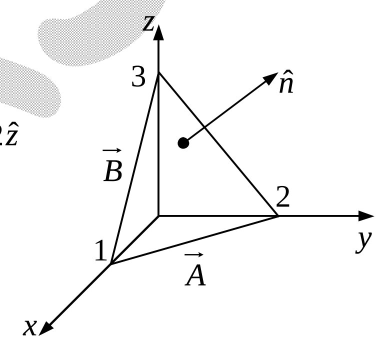
Ans. 2: (a)

Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$$\vec{A}=-\hat{x}+2\hat{y};\quad \vec{B}=-\hat{x}+3\hat{z}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ -1 & 2 & 0\\ -1 & 0 & 3\end{vmatrix}=\hat{x}(6-0)+\hat{y}(-3+0)+\hat{z}(0+2)=6\hat{x}+3\hat{y}+2\hat{z}$$

$$\Rightarrow\hat{n}=\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}=\frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$$



Ans. 3: (b)

Solution: To get a normal at the surface let's take the gradient

$$\vec{\nabla}(xyz)=yz\hat{i}+zx\hat{j}+\hat{k}xy=8\hat{i}+4\hat{j}+2\hat{k}$$

We want a plane perpendicular to this so: $\left(\vec{r}-\vec{r}_0\right)\cdot\frac{\left(8\hat{i}+4\hat{j}+2\hat{k}\right)}{\sqrt{64+16+4}}=0$.

$$\left[(x-1)\hat{i}+(y-2)\hat{j}+(z-4)\hat{k}\right]\cdot\left[8\hat{i}+4\hat{j}+2\hat{k}\right]=0\Rightarrow 4x+2y+z=12.$$

Ans. 4: (c)

Solution: Let $\phi=x+y+z-5\ \Rightarrow\vec{\nabla}\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z-5)=\hat{i}+\hat{j}+\hat{k}$

Ans. 5: (a)

Solution: Here $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$. Unit normal vector is $\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$.

$$\text{So, } \vec{\nabla}\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = \frac{2x\hat{i}}{a^2} + \frac{2y\hat{j}}{b^2} + \frac{2z\hat{k}}{c^2}$$

$$\vec{\nabla}\phi \Big|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}$$

$$\left|\vec{\nabla}\phi\right| = \sqrt{\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{4}{3c^2}} = \frac{2}{\sqrt{3}}\sqrt{\frac{b^2c^2 + a^2c^2 + a^2c^2}{a^2b^2c^2}}$$

$$\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}\Bigg|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{\frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}}{\frac{2}{\sqrt{3}}\frac{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}{abc}} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

Ans. 6: (b)

Solution: $z = \pm\sqrt{\frac{3}{2}x^2 + \frac{3}{2}y^2} \Rightarrow z^2 = \frac{3}{2}x^2 + \frac{3}{2}y^2 \Rightarrow 3x^2 + 3y^2 - 2z^2 = 0$

Let $V = 3x^2 + 3y^2 - 2z^2$, Taking gradient $\Rightarrow \vec{\nabla}V = 6x\hat{x} + 6y\hat{y} - 4z\hat{z}$.

The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}}, 0, 1\right)$ is $\hat{n} = \frac{\vec{\nabla}V}{\left|\vec{\nabla}V\right|}$. Thus

$$\hat{n} = \frac{6\sqrt{\frac{2}{3}}\hat{x} + 6\times 0\hat{y} - 4\times 1\hat{z}}{\sqrt{36\times\frac{2}{3} + 16}} = \frac{6\sqrt{\frac{2}{3}}\hat{x} - 4\hat{z}}{\sqrt{40}} = \sqrt{\frac{3}{5}}\hat{x} - \frac{2}{\sqrt{10}}\hat{z}$$

Ans. 7: (d)

Solution: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \hat{x}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{y}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{z}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

Ans. 8: (d)

Solution: $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$\because A_r = 10,\ \ A_\theta = 5\sin\theta,\ \ A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times 10\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\times 5\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial(0)}{\partial\phi}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = \left(2+\cos\theta\right)\left(10/r\right)$$

Ans. 9: (a)

Solution: $\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right) + \frac{1}{r}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \Rightarrow \vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{150}{r^2}\right) + \frac{1}{r}\frac{\partial(10)}{\partial\phi} + \frac{\partial(0)}{\partial z} = \frac{-150}{r^3}$

Ans. 10: (b)

Solution: $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & e^{-r} & 0 \end{vmatrix} \qquad \because A_r = 0,\ A_\theta = \left(e^{-r}/r\right),\ A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\times r\sin\theta\hat{\phi}\left(\frac{\partial e^{-r}}{\partial r}\right) = -\frac{e^{-r}}{r}\hat{\phi}$$

Ans. 11: (d)

Solution:

$$\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & rk & 0\end{pmatrix}=\frac{k}{r}\hat{z}\Rightarrow\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\frac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & 0 & k/r\end{pmatrix}=-\frac{r\hat{\phi}}{r}\left(-\frac{k}{r^2}\right)=\frac{k}{r^2}\hat{\phi}$$

Ans. 12: (a)

Solution: Since $\vec{F}$ is derivative from potential $V(r)$ and $\vec{F}=-\vec{\nabla}V(r)$

$$\Rightarrow\vec{\nabla}\times\vec{F}=-\vec{\nabla}\times\left(\vec{\nabla}V\right)=0.$$

Ans. 13: (b)

Solution: Let $\vec{F}=F_0\left(\hat{x}+\hat{y}+\hat{z}\right)$ and $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}\Rightarrow\vec{F}.\vec{r}=F_0\left(x+y+z\right)$.

$$\text{Thus }\vec{\nabla}\left(\vec{F}\cdot\vec{r}\right)=F_0\left(\hat{x}+\hat{y}+\hat{z}\right)=\vec{F}$$

Ans. 14: (d)

Solution: Let $\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$, $\vec{B}=B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$ and $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$.

$$\vec{B}\times\vec{r}=\left(B_yz-yB_z\right)\hat{x}+\left(B_zx-zB_x\right)\hat{y}+\left(B_xy-xB_y\right)\hat{z}$$

$$\Rightarrow\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=A_x\left(B_yz-yB_z\right)+A_y\left(B_zx-zB_x\right)+A_z\left(B_xy-xB_y\right)$$

$$\Rightarrow\vec{\nabla}\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}=\vec{A}\times\vec{B}$$

Ans. 15: (d)

Solution: $\oint_C\vec{A}\cdot d\vec{l}=\int_S\left(\vec{\nabla}\times\vec{A}\right)d\vec{a}=0\quad\because\vec{\nabla}\times\vec{A}=0$

Ans. 16: (b)

Solution: It is given that $\oint_S\vec{A}.d\vec{a}=0\Rightarrow\int_V\left(\vec{\nabla}.\vec{A}\right)d\tau=0$ (From Divergence Theorem)

$$\int_V\left(\vec{\nabla}.\vec{A}\right)d\tau=0\Rightarrow 2k+l-3m=0$$

Ans. 17: (a)

Solution: Since $\vec{A} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k} \Rightarrow \vec{\nabla}.\vec{A} = z^2 - z^2 + \left(x^2 - y^2\right) = x^2 - y^2$

Thus

$$\int_V \left(\vec{\nabla}.\vec{A}\right)d\tau = \int_{x=-a}^{x=+a}\int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left(x^2 - y^2\right)dxdydz = \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{x^3}{3} - y^2x\right]_{-a}^{+a} dydz = \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{2}{3}a^3 - 2ay^2\right]dydz$$

$$\Rightarrow \int_V \left(\vec{\nabla}.\vec{A}\right)d\tau = \int_{z=0}^{z=c}\left[\frac{2}{3}a^3y - 2a\frac{y^3}{3}\right]_{-b}^{+b} dz = \int_{z=0}^{z=c}\left[\frac{4}{3}a^3b - \frac{4}{3}ab^3\right]dz = \frac{4}{3}abc\left[a^2 - b^2\right]$$

Ans. 18: (b)

Solution: $\vec{v} = \vec{\nabla}\times\vec{u} \Rightarrow \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = 0, \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = 0, \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = v_0$

Let $u_x = 0, u_z = 0 \Rightarrow \vec{u} = v_0 x\hat{j}$

Ans. 19: (b)

Solution: Thus $\int_S \vec{v}.d\vec{a} = \int_S \left(\vec{\nabla}\times\vec{u}\right).d\vec{a} = \oint_{line} \vec{u}.d\vec{l}$

We have to take line integral around circle $x^2 + y^2 = r^2$ in $z = 0$ plane. Let use cylindrical coordinate and use $x = r\cos\phi,\ y = r\sin\phi \Rightarrow dy = r\cos\phi d\phi$.

$$\Rightarrow \oint_{line} \vec{u}.d\vec{l} = \oint_{line} v_0 x dy = \int_0^{2\pi} v_0 \times r\cos\phi \times r\cos\phi d\phi$$

$$\Rightarrow \oint_{line} \vec{u}.d\vec{l} = v_0 r^2\int_0^{2\pi}\cos^2\phi d\phi = v_0 r^2\int_0^{2\pi}\left[\frac{1+\cos 2\phi}{2}\right]d\phi = \pi v_0 r^2$$

$$\Rightarrow \int_S \vec{v}.d\vec{a} = \int_S \left(\vec{\nabla}\times\vec{u}\right).d\vec{a} = \oint_{line} \vec{u}.d\vec{l} = \pi v_0 r^2$$

Ans. 20: (c)

Solution: $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & 2xy & 0 \end{vmatrix} = 0$

Thus the force $\vec{F}(x, y) = k\left[\left(x^2 + y^2\right)\hat{i} + 2xy\ \hat{j}\right]$ is conservative. So work done is independent of paths.

Along line OD, $y = x \Rightarrow dy = dx$

Since $d\vec{l} = dx\hat{i} + dy\hat{j} \Rightarrow \vec{F}.d\vec{l} = \left[\left(x^2 + y^2\right)dx + 2xy\, dy\right] = \left[\left(x^2 + x^2\right)dx + 2x^2 dx\right] = 4x^2 dx$

$$\int_{OD} \vec{F}.d\vec{l} = \int_{x=0}^{1} 4x^2 dx = \frac{4}{3}$$

Ans. 21: (a)

Solution: Since $\vec{\nabla} \times \vec{F} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{pmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2y + 2y) = 4y\hat{k}$

Using the Stokes' theorem

$$\oint_{OPQRO} \vec{F}.d\vec{l} = \int_S \left(\vec{\nabla} \times \vec{F}\right).d\vec{a} = \int_0^a \int_0^b \left(4y\hat{k}\right).\left(-dxdy\hat{k}\right) = -4\int_0^a \int_0^b ydxdy = -4a\int_0^b ydy = -2ab^2$$

Ans. 22: (a)

Solution: For an arbitrary test function $f(x)$, consider the integral $\int_{-\infty}^{\infty} f(x)\delta(kx)dx$.

Changing variables, we let $y = kx$, so that $x = y/k\, x$, and $dx = \frac{1}{k}dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is negative, then $x = \infty$ implies $y = -\infty$, and vice versa, so the order of limit is reversed. Restoring the "proper" order costs a minus sign. Thus

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k} = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0)$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k, as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx \Rightarrow \delta(kx) = (1/|k|)\delta(x)$$

Ans. 23: (a)

Solution: Answer would be 0, because the spike would then be outside the domain of integration.

Ans. 24: (d)

Solution: $$J = \int_v (r^2+2)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_v (r^2+2)4\pi\delta^3(r)\,d\tau = 4\pi(0+2) = 8\pi$$

Ans. 25: (c)

Solution: The delta function picks out the value of x^3 at the point $x=2$ so the integral is $2^3 = 8$.

NAT (Numerical Answer Type)

Ans. 26: 0

Solution: $x^2+y^2=1 \Rightarrow xdx = -ydy$ and $d\vec{l} = dx\hat{x} + dy\hat{y}$

$$\Rightarrow \vec{F}.d\vec{l} = \frac{xdx}{\sqrt{x^2+y^2}} + \frac{ydy}{\sqrt{x^2+y^2}} = 0 \quad (\because xdx = -ydy) \Rightarrow \int_A^B \vec{F}\cdot d\vec{l} = 0$$

Ans. 27: 3

Solution: $$\oint_S \vec{r}.\hat{n}ds = \int_V (\vec{\nabla}.\vec{r})d\tau = 3\int_V d\tau = 3\pi a^2 h$$

Ans. 28: 4

Solution: In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$$\because A_r = 2r\cos^2\phi,\quad A_\phi = 3r^2\sin z,\quad A_z = 4z\sin^2\phi$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r\times 2r\cos^2\phi) + \frac{1}{r}\frac{\partial(3r^2\sin z)}{\partial\phi} + \frac{\partial(4z\sin^2\phi)}{\partial z}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4(\cos^2\phi + \sin^2\phi) = 4$$

Ans. 29: 0.5

Solution: $\vec{u} = a\hat{i} + b\hat{n}$, $\vec{v} = a\hat{n} + b\hat{i}$

$$\Rightarrow \vec{u}\cdot\vec{v} = \left(a\hat{i} + b\hat{n}\right)\cdot\left(a\hat{n} + b\hat{i}\right) \Rightarrow |\vec{u}||\vec{v}|\cos 60 = a^2\hat{i}\cdot\hat{n} + ab + ba + b^2\hat{n}.\hat{i}$$

$$\left(\sqrt{a^2 + b^2 + 2ab\cos 120}\right)^2 \cdot \cos 60 = a^2\cos 120 + 2ab + b^2\cos 120$$

$$\left(a^2 + b^2 - 2ab\times\frac{1}{2}\right)\cdot\cos 60 = -\frac{1}{2}\left(a^2 + b^2\right) + 2ab = \frac{1}{2}\left(a^2 + b^2\right) - \frac{ab}{2} = -\frac{1}{2}\left(a^2 + b^2\right) + 2ab$$

$$\Rightarrow a^2 + b^2 = \frac{5ab}{2} \Rightarrow b = \frac{a}{2}$$

Ans. 30 : 3

Solution : Since $\oint_S \vec{A}.d\vec{a} = \int_V \left(\vec{\nabla}.\vec{A}\right)d\tau \Rightarrow \int_V \left(\vec{\nabla}.\vec{r}\right)d\tau = 3V$

MSQ (Multiple Select Questions)

Ans. 31: (a), (b) and (d)

Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x}$, $y = 1$, $\vec{A}\cdot d\vec{l} = y^2 dx = dx$, so $\int \vec{A}\cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y}$, $x = 2$, $\vec{A}\cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy$, so $\int \vec{A}\cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$.

By path (1), then, $\int_a^b \vec{A}\cdot d\vec{l} = 1 + 10 = 11$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y},\quad \vec{A}\cdot d\vec{l} = x^2 dx + 2x(x+1)dx = \left(3x^2 + 2x\right)dx$$

so $$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 \left(3x^2 + 2x\right)dx = \left(x^3 + x^2\right)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then, $\oint \vec{A}\cdot d\vec{l} = 11 - 10 = 1$